Contributing to the Formalization of Carleson's Theorem

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Geboren am 2. Februar 2004 in Viersen

29. Juli 2024

Bachelorarbeit Mathematik

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Abstract

This bachelor thesis is a contribution to the ongoing collective effort of formalizing Carleson's theorem about pointwise convergence of Fourier series. Using the Lean Theorem Prover, this project aims to verify a recent result about estimates for a generalized Carleson operator and the deduction of Carleson's classical theorem from this result. Here, we work towards the deduction step. As part of this thesis, the main line of arguments connecting the two theorems has been formalized in Lean, building on its extensive mathematical library mathlib. We give an outline of the tools used, the challenges encountered and the design choices made during the course of working towards the deduction's formalization.

However, to fully finish the deduction, one needs to verify an assumption of the general theorem on the boundedness of a rather involved operator, albeit one that can be handled using classical Calderón–Zygmund theory. This involves as a major step reducing the assumption to bounding a family of simpler operators which on the real line are truncated Hilbert transforms.

It turns out that the reduction step can and should be performed in the general setting, requiring only slightly stronger regularity assumptions, namely a two-sided instead of a one-sided Calderón–Zygmund kernel. Carrying out this generalization (in the informal setting, preparing future formalization) is the other major part of this thesis. As a result, a new variant of the general theorem for a two-sided Calderón–Zygmund kernel with weakened assumption on operator bounds is stated, simplifying applications of the general result.

Zusammenfassung in deutscher Sprache

Die vorliegende Bachelorarbeit stellt einen Beitrag zu dem laufenden Gemeinschaftsprojekt dar, den Satz von Carleson über punktweise Konvergenz von Fourier-Reihen zu formalisieren. Mithilfe des Lean Theorem Prover sollen dabei ein neues Ergebnis über Abschätzungen für einen verallgemeinerten Carleson-Operator sowie die Folgerung des klassischen Satzes von Carleson aus diesem Ergebnis verifiziert werden. Die vorliegende Arbeit beschäftigt sich mit den für die Deduktion erforderlichen Schritten. Ein wesentlicher Bestandteil der Arbeit ist die Formalisierung der Hauptargumentationslinie, die die beiden Sätze verbindet. Dies erfolgt in Lean, aufbauend auf dessen umfangreicher Mathematik-Bibliothek mathlib. Es wird ein Überblick über Designentscheidungen, verwendete Hilfsmittel sowie Herausforderungen gegeben.

Um die Deduktion abzuschließen, muss jedoch noch eine Annahme des allgemeinen Satzes verifiziert werden, bezüglich einer Schranke für einen komplizierten, aber mithilfe klassischer Calderón–Zygmund-Theorie behan-

delbaren Operator. Dabei wird die Annahme zunächst auf Beschränktheitsaussagen für eine Familie einfacherer Operatoren reduziert, die auf dem Raum der reellen Zahlen abgeschnittene Hilberttransformationen sind.

Es stellt sich heraus, dass es möglich und sinnvoll ist, den Reduktionsschritt unter leicht stärkeren Annahmen an die Regularität des Calderón-Zygmund-Kerns im allgemeinen Rahmen durchzuführen. Die Ausarbeitung dieser Verallgemeinerung (noch informell, als Vorbereitung für eine zukünftige Formalisierung) stellt den anderen großen Teil dieser Arbeit dar. Als Ergebnis wird eine neue Variante des allgemeinen Satzes für einen zweiseitigen statt einseitigen Calderón–Zygmund-Kern und dafür abgeschwächter Annahme an Schranken für Operatoren formuliert, was die Anwendung des Satzes vereinfacht.

Contents

Acknowledgements

I would like to thank my first advisor Prof. Dr. Floris van Doorn for his continuous support in all matters of formalization and beyond, always providing incredibly helpful feedback. In particular, whenever I got stuck with Lean and started to blame the computer for it, he showed me how it was my fault and how to do things the right way. As manager of the Carleson project, he was always quick to integrate new work into the project, encouraging continuous improvement.

I would also like to thank Prof. Dr. Christoph Thiele for encouraging me to contribute to the mathematical side of the project, and for answering all questions tightly or broadly related with great patience, making me feel like having a second first advisor.

Of the international contributors of the Carleson project, I would like to highlight the help of Pietro Monticone who has cleaned up some of my code when it was not using some of Lean's features to make it more concise or when it did not comply with best practice guidelines.

Finally, I would like to emphasize that I am very grateful to both of my advisors for the great opportunity of participating in this exciting, topical and international math project.

1 Introduction

The author's task was to work on the formalization of chapter 10 of [Bec+24], i.e. the deduction of Carleson's classical theorem from the metric space Carleson theorem, which proves extimates for a generalized Carleson operator. It is part of the Carleson project aiming to formalize all of [Bec+24] in Lean. The project's homepage and github repository can be found under
`https://florisvandoorn.com/carleson/`
and
`https://github.com/fpvandoorn/carleson`.

The underlying paper [Bec+24] provides more details than usual in this field to serve as a largely self-contained blueprint for the formalization process. It also evolves along this process, as doing formalization naturally encourages to consider worthwhile generalizations, to find shortcuts building on the contents of the mathematical library, to split long proofs into independent smaller parts, to fill in (even relatively small) gaps, and to detect and correct minor errors.

The author's relevant contributions here, in particular the formulation and proof of a new variant of the metric space Carleson theorem, are described in Section 2.1. The actual result of this work is included as an excerpt from the current version of the blueprint in Section 3.

The progress made in the formalization is overviewed in Section 2.2 and the corresponding code, attached to this document, constitutes a main part of this thesis. A more detailed presentation of design choices, formalized statements and the challenges encountered during formalizing them and their proofs can be found in Section 4.

We begin by briefly introducing Carleson's classical theorem in Section 1.1 and the general result it will follow from in Section 1.2. Since we focus on the contributions to the formalization in this thesis, we do not attempt to present the history and variants of Carleson's theorem and its proof, instead referring to [Dem12].

We complement the introduction with a short survey of Lean and mathlib.

1.1 Carleson's Theorem

For a complex valued function $f \in L^2([0, 2\pi])$ and $n \in \mathbb{Z}$, define the n-th Fourier coefficient as

$$\widehat{f_n} := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx. \tag{1.1}$$

We define the N-th partial Fourier sum for an integer $N \geq 0$ as

$$S_N f(x) := \sum_{n=-N}^{N} \widehat{f_n} e^{inx} \; . \tag{1.2}$$

In [Car66], L. Carleson proved that for $f \in L^2([0, 2\pi])$ we have that for almost every $x \in [0, 2\pi]$,

$$S_N f(x) \to f(x)$$

as $N \to \infty$.

For simplicity, we restrict here to the case of continuous functions. As explained in [Bec+24], for the almost everywhere convergence result it suffices to show the following, in which we allow an exceptional set of positive measure.

Theorem 1.1 (classical Carleson)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and continuous. For all $\epsilon > 0$, there exists a Borel set $E \subset [0, 2\pi]$ with Lebesgue measure $|E| \leq \epsilon$ and a positive integer N_0 such that for all $x \in [0, 2\pi] \setminus E$ and all integers $N > N_0$, we have*

$$|f(x) - S_N f(x)| \leq \epsilon. \tag{1.3}$$

1.2 Estimates for the generalized Carleson operator

The proof of Theorem 1.1 relies on Theorem 1.2. In the following, we include the most important definitions required to state this theorem. For the here less important definition of a cancellative compatible collection of functions on a doubling metric measure space, we refer to [Bec+24].

The remainder of this subsection has been taken unchanged from the blueprint respectively [Bec+24].

A doubling metric measure space (X, ρ, μ, a) is a complete and locally compact metric space (X, ρ) equipped with a σ-finite non-zero Radon–Borel measure μ that satisfies the doubling condition that for all $x \in X$ and all $R > 0$ we have

$$\mu(B(x, 2R)) \leq 2^a \mu(B(x, R))\,, \tag{1.4}$$

where we have denoted by $B(x, R)$ the open ball of radius R centred at x:

$$B(x, R) := \{y \in X : \rho(x, y) < R\}. \tag{1.5}$$

A one-sided Calderón–Zygmund kernel K on the doubling metric measure space (X, ρ, μ, a) is a measurable function

$$K : X \times X \to \mathbb{C} \tag{1.6}$$

such that for all $x, y', y \in X$ with $x \neq y$, we have

$$|K(x, y)| \leq \frac{2^{a^3}}{V(x, y)} \tag{1.7}$$

and if $2\rho(y, y') \leq \rho(x, y)$, then

$$|K(x, y) - K(x, y')| \leq \left(\frac{\rho(y, y')}{\rho(x, y)}\right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x, y)}, \tag{1.8}$$

where

$$V(x, y) := \mu(B(x, \rho(x, y))).$$

Define the maximally truncated non-tangential singular integral T_* associated with K by

$$T_* f(x) := \sup_{R_1 < R_2} \sup_{\rho(x,x') < R_1} \left| \int_{R_1 < \rho(x',y) < R_2} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| . \tag{1.9}$$

For a cancellative compatible collection of functions Θ, we define the generalized Carleson operator T by

$$T f(x) := \sup_{\vartheta \in \Theta} \sup_{0 < R_1 < R_2} \left| \int_{R_1 < \rho(x,y) < R_2} K(x, y) f(y) e^{i\vartheta(y)} \, \mathrm{d}\mu(y) \right| . \tag{1.10}$$

Theorem 1.2 (metric space Carleson)**.** *For all integers $a \geq 4$ and real numbers $1 < q \leq 2$ the following holds. Let (X, ρ, μ, a) be a doubling metric measure space. Let Θ be a cancellative compatible collection of functions and let K be a one-sided Calderón–Zygmund kernel on (X, ρ, μ, a). Assume that for every bounded measurable function g on X supported on a set of finite measure we have*

$$\|T_* g\|_2 \leq 2^{a^3} \|g\|_2 \,, \tag{1.11}$$

where T_* is defined in (1.9)*. Then for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have, with T defined in* (1.10)*,*

$$\left| \int_G T f \, \mathrm{d}\mu \right| \leq \frac{2^{450 a^3}}{(q-1)^6} \mu(G)^{1 - \frac{1}{q}} \mu(F)^{\frac{1}{q}} \,. \tag{1.12}$$

1.3 Lean and mathlib

Lean is an interactive theorem prover that allows users to write mathematical definitions, statements and proofs in a formal language and have their proofs checked automatically. The underlying logical framework is built on dependent types, based on the calculus of inductive constructions [MU21; Com20].

Formalizing mathematics in general has the benefit of ensuring precision of statements and correctness of proofs. Convincing human mathematicians to trust a result that has been formalized reduces to the tasks of arguing for consistency of the formal definitions with the informal ones, and building trust in the correctness of the verifying program. On the other hand, building a formalization will generally take much longer than writing down the informal proof.

One key to make formalizing research level mathematics feasible on a larger scale are libraries of formalized mathematics covering foundational theories of many different fields. With mathlib, Lean has a fast growing mathematical library that has already enabled a number of recent and complicated results to be formalized [Com20]. The Carleson project, for example, heavily builds on measure theory in mathlib, including the related notions of the Bochner integral and the lower Lebesgue integral.

Another very important aspect of Lean and mathlib is the automation provided by so called tactics. They can perform various general and more specific tasks, from rewriting expressions using explicitly given lemmas to transforming expressions within an algebraic structure into a certain normal form or proving linear inequalities from hypotheses present in the context [Com20]. Their use improves both the speed of writing code and, in many instances, its readibility by shortening trivial or purely calculational sections of a proof.

Further support is available in the form of linters that can hint at various (suspected) issues with the code that are not formal errors but might still be unintended or cause problems at a later point in time, and an extensive online documentation [DEL20].

2 Overview of contributions

2.1 Mathematical contribution

The direct path from Theorem 1.2 to Theorem 1.1 starts with an approximation argument similar to the one in the Preliminaries of [Fef73] to reduce Theorem 1.1 to Lemma 3.40, which uniformly bounds the sup norm of all partial Fourier sums of a function in terms of the sup norm of the function, outside of some exceptional set. Its proof uses Lemma 3.25, a specialization of Theorem 1.2 for the real line. While many of the assumptions of Theorem 1.2 are relatively straightforward in this special case, the assumption (1.11) on the operator $T*$ is more difficult to obtain.

In [Bec+24], assumption (1.11) is verified by first reducing it to bounding a much simpler family of operators (in 10.3 and 10.5) and then proving these bounds (in 10.4). A careful analysis of the reduction reveals the properties of the kernel $K(x, y) := \kappa(x-y)$ it relies on: The decay and regularity property of a general one-sided Calderón–Zygmund kernel, as well as the symmetry due to its convolution form. It turns out, however, that the symmetry's only relevant consequence for the reduction is that we also have regularity in the first variable, i.e. a two-sided Calderón–Zygmund kernel. Since this additional regularity assumption might hold in many other applications, this motivates to do the reduction in the general setting. Its result is now stated as Theorem 3.2.

Generalizing the proof required the following: For 10.3, which corresponds to Section 3.1.1 here, most statements and proofs could be translated without changing their structure. However, in contrast to the real setting, it was necessary to pay attention to the boundary of balls since they may have non-zero measure in general doubling metric measure spaces. Lemma 3.11 and Lemma 3.12 have been added to prove that the integral in the definition of Tf can be exchanged for an integral over the difference of open balls.

In 10.5, which corresponds to Section 3.1.2 here, a much more general version of Calderón-Zygmund decomposition was required. Since the original blueprint proof was relatively specific to the one-dimensional case, the generalization follows a different path taken from [Ste93]. The underlying sections of the book are referenced in more detail in Section 3.1.2. In the generalized Calderón-Zygmund decomposition in Lemma 3.17, we lose disjointedness of the intervals I_j respectively the balls B_j^* in the general setting. As a substitute, we state and prove a bounded intersection property of the B_j^* which is used in the proof of Lemma 3.4. Other small additions had to be made to account for the special case of $\mu(X) < \infty$ in the underlying doubling metric measure space. The proof of Lemma 3.4 has been split into multiple smaller parts to make it more readable and easier to formalize.

In both Sections 3.1.1 and 3.1.2, many constants depending on the parameter a had to be worked out.

In the following, we will list in detail smaller changes and additions made to the corresponding sections of the blueprint, compared to the reference version [Bec+24].

The author claims no originality apart from where it is explicitly mentioned, and indeed all contributions, including the ones to Sections 3.1.1 and 3.1.2, should be understood as edits to an existing document, with many statements, sentences, equations and proof structures remaining unchanged.

- The approximation argument carried out in 10.1 of [Bec+24] relies on a class of functions with the properties that they can approximate a continuous function in the sup norm, and that uniform convergence of their Fourier series outside of a small exceptional set can be shown easily. Originally, this class was chosen to be piecewise constant functions with uniform piece size. However, due in particular to the results already present in mathlib on Fourier series, it turned out to be much more efficient to use smooth functions instead. Since the Fourier series of smooth functions converges uniformly on the whole of $[0, 2\pi]$, eliminating the need for a second exceptional set, the approximation argument could be simplified further. Its current version can be found in Section 3.2.1 and Section 3.2.2.

- The statement of Lemma 3.27 has been corrected to have its definition of the Lipschitz norm match the specialized definition in Section 3.2.7. The assumptions on α and β have been weakened. Its proof has been supplemented to deal with the cases $n = 0$, $n < 0$ and $\beta - \alpha < \frac{\pi}{n}$ explicitly.

- The proof in 10.8 of [Bec+24] has been split into smaller parts in Section 3.2.6. The intermediate, partly generalized statements, in the author's opinion, are not only helpful for formalization but also for clarifying the central ideas behind the transition from estimating $S_N g$ to estimating Tg. There has also been added an explicit proof of the measurability of the Carleson operator of a function in the real setting, in Lemma 3.39. Such measurability proofs are likely to have been obviated in other parts of the blueprint as well. When they become inevitable during the formalization process, they might be modeled after this proof and its corresponding formalized version.

- The constants in Lemma 3.32 and Lemma 3.33 have been improved and in the proof of Lemma 3.33, a minor error in the third case of the case distinction has been fixed.

2.2 Formalization results

The structure of a mathematical theory consisting of many different statements and their interdependencies forms an acyclic directed graph. This dependency graph can be visualized and it also provides an opportunity to track and illustrate the progress of formalization.

Figure 1 shows the dependency graph of the Carleson project at the time of writing.[1] A green background of a node indicates that the proof of the corresponding theorem or lemma has been formalized. Dark green means full verification while light green means conditional verification that still depends directly or indirectly on a result not yet verified. A blue background indicates that all prerequisites of a result are done and it is ready to turn dark green once its proof is formalized. The distinct group of nodes at the bottom of Figure 1 is displayed in more detail in Figure 2. Apart from the nodes leading to `two-sided-metric-space-Carleson`, it corresponds to Section 3.2. The green nodes there have been contributed to the formalization by the author and they comprise Section 3.2 except for Section 3.2.3 and Section 3.2.5. In [Bec+24], this broadly corresponds to 10.1, 10.2, 10.6, 10.8 and 10.9, although some of these are no longer up to date, see Section 2.1.

Much of is Figure 2 is centered around `real-Carleson`. This is Lemma 3.25, a version of Theorem 1.2 for the real line, and proving it means to verify the assumptions in this concrete case, which is done in most of its predecessors in the dependency graph. The nodes on the same level or below it on the other hand correspond to the approximation argument that really shows how Theorem 1.1 follows from the estimate for the generalized Carleson operator. As it can be seen in Figure 2, this line of arguments is fully formalized modulo the formalization of the remaining prerequisites of `real-Carlson`.

[1] See `https://florisvandoorn.com/carleson/blueprint/dep_graph_document.html` for the current version.

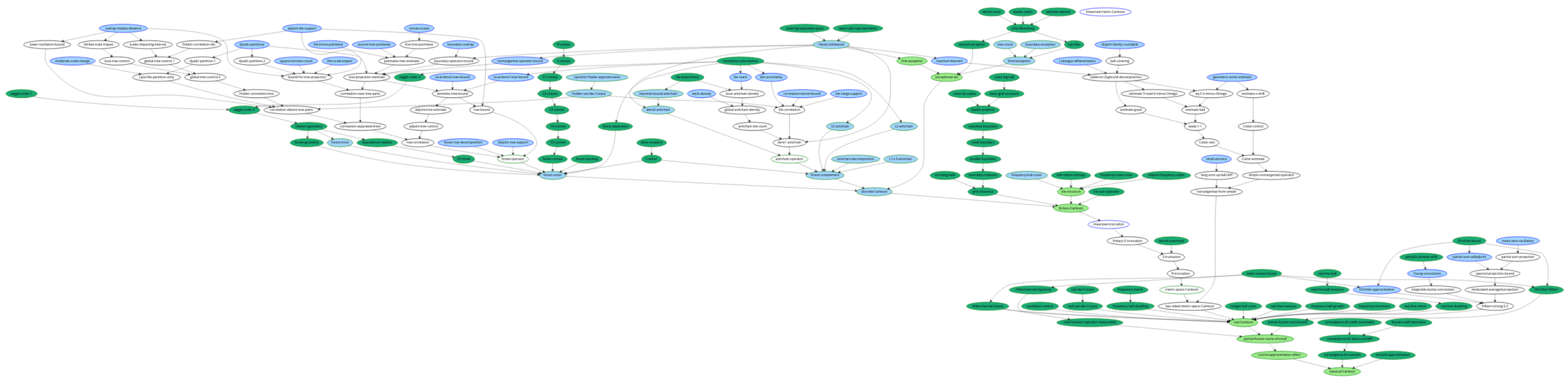

Figure 1: Dependency graph of the Carleson project

Figure 2: Dependency graph of the Carleson project, close-up of the nodes corresponding to the deduction

3 Blueprint excerpts

Preliminaries

The following excerpts are mostly independent from other parts of the blueprint. To avoid undefined references for the one theorem that is cited multiple times, we redirect them here.

Theorem 3.1 (Hardy-Littlewood)**.** *This is Proposition 2.6 in [Bec+24]. Note however that there is a typo in this version and Equation (2.46) there actually holds for* $1 \le p_1 < p_2$*, just like Equation (2.44).*

3.1 Two-sided Metric Space Carleson

We prove a variant of Theorem 1.2 for a two-sided Calderón–Zygmund kernel on the doubling metric measure space (X, ρ, μ, a), i.e. a one-sided Calderón–Zygmund kernel K which additionally satisfies for all $x, x', y \in X$ with $x \neq y$ and $2\rho(x, x') \le \rho(x, y)$,

$$|K(x,y) - K(x',y)| \le \left(\frac{\rho(x,x')}{\rho(x,y)}\right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x,y)}. \tag{3.1}$$

By the additional regularity, we can weaken the assumption (1.11) to a family of operators that is easier to work with in applications. Namely, for $r > 0$, $x \in X$, and a bounded, measurable function $f : X \to \mathbb{C}$ supported on a set of finite measure, we define

$$T_r f(x) := \int_{r \le \rho(x,y)} K(x,y) f(y)\, d\mu(y) = \int_{X \setminus B(x,r)} K(x,y) f(y)\, d\mu(y). \tag{3.2}$$

Theorem 3.2 (two-sided metric space Carleson)**.** *For all integers* $a \ge 4$ *and real numbers* $1 < q \le 2$ *the following holds. Let* (X, ρ, μ, a) *be a doubling metric measure space. Let* Θ *be a cancellative compatible collection of functions and let* K *be a two-sided Calderón–Zygmund kernel on* (X, ρ, μ, a)*. Assume that for every bounded measurable function* g *on* X *supported on a set of finite measure and all* $r > 0$ *we have*

$$\|T_r g\|_2 \le 2^{a^3} \|g\|_2 \,. \tag{3.3}$$

Then for all Borel sets F *and* G *in* X *and all Borel functions* $f : X \to \mathbb{C}$ *with* $|f| \le \mathbf{1}_F$*, we have, with* T *defined in* (1.10),

$$\left| \int_G T f \, \mathrm{d}\mu \right| \le \frac{2^{452a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}} \,. \tag{3.4}$$

For the remainder of this chapter, fix an integer $a \ge 4$, a doubling metric measure space (X, ρ, μ, a) and a two-sided Calderón–Zygmund kernel K as in Theorem 3.2.

The following lemma is proved in Section 3.1.1.

Lemma 3.3 (nontangential-from-simple)**.** *Assume* (3.3) *holds. Then, for every bounded measurable function* $g : X \to \mathbb{C}$ *supported on a set of finite measure we have*

$$\|T_* g\|_2 \leq 2^{3a^3} \|g\|_2. \tag{3.5}$$

Proof of Theorem 3.2. Let $1 < q \leq 2$ be a real number. Let Θ be a cancellative compatible collection of functions. By the assumption (3.3), we can apply Lemma 3.3 to obtain for every bounded measurable $g : X \to \mathbb{C}$ supported on a set of finite measure,

$$\|T_* g\|_2 \leq 2^{3a^3} \|g\|_2. \tag{3.6}$$

Define

$$K'(x, y) := 2^{-2a^3} K(x, y)\,.$$

Then K' is a two-sided Calderón–Zygmund kernel on (X, ρ, μ, a). Denote the corresponding maximally truncated non-tangential singular operator by T'_* and the corresponding generalized Carleson operator by T'. With (3.6), we obtain for g as above,

$$\|T'_* g\|_2 \leq 2^{a^3} \|g\|_2. \tag{3.7}$$

Applying Theorem 1.2 for K' yields that for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have

$$\left| \int_G T' f \,\mathrm{d}\mu \right| \leq \frac{2^{450a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}\,.$$

This finishes the proof since for all $x \in X$,

$$T' f(x) = 2^{-2a^3} T f(x)\,.$$

□

The proof of Lemma 3.3 relies on the following auxiliary lemma which is proved in Section 3.1.2.

Lemma 3.4 (Weak 1 1)**.** *Let* $f : X \to \mathbb{C}$ *be a bounded measurable function supported on a set of finite measure and assume for some* $r > 0$ *that for every bounded measurable function* $g : X \to \mathbb{C}$ *supported on a set of finite measure,*

$$\|T_r g\|_2 \leq 2^{a^3} \|g\|_2. \tag{3.8}$$

Then for all $\alpha > 0$*, we have*

$$\mu\left(\{x \in X : |T_r f(x)| > \alpha\}\right) \leq \frac{2^{a^3+19a}}{\alpha} \int |f(y)|\, d\mu(y). \tag{3.9}$$

Throughout Sections 3.1.1 and 3.1.2, for any measurable bounded function $w : X \to \mathbb{C}$, let $Mw : X \to [0, \infty)$ denote the corresponding Hardy–Littlewood maximal function defined in Theorem 3.1. Apart from Theorem 3.1, Sections 3.1.1 and 3.1.2 have no dependencies in the previous chapters.

3.1.1 Proof of Cotlar's Inequality

Lemma 3.5. *For all real numbers $x \geq 4$,*

$$\sum_{n=0}^{\infty} 2^{-\frac{n}{x}} \leq 2^x.$$

Proof. By convexity, for all $0 \leq \lambda \leq 1$

$$2^{\lambda(-\frac{1}{4})} \leq \lambda 2^{-\frac{1}{4}} + (1-\lambda)2^0.$$

For $\lambda := \frac{4}{x}$, we obtain

$$2^{-\frac{1}{x}} \leq 1 - (1 - 2^{-\frac{1}{4}})\frac{4}{x}.$$

We conclude

$$\sum_{n=0}^{\infty} 2^{-\frac{n}{x}} = \frac{1}{1 - 2^{-\frac{1}{x}}} \leq \frac{1}{4(1 - 2^{-\frac{1}{4}})} x \leq 2^x.$$

□

Lemma 3.6 (estimate x shift)**.** *Let $0 < r$ and $x \in X$. Let $g : X \to \mathbb{C}$ be a bounded measurable function supported on a set of finite measure. Then for all x' with $\rho(x, x') \leq r$.*

$$\left|T_r g(x) - T_r g(x')\right| \leq 2^{a^3+2a+2} M g(x)\,.$$

Proof. By definition, the right hand side above is equal to

$$\left| \int_{r \leq \rho(x,y)} K(x,y) g(y)\, d\mu(y) - \int_{r \leq \rho(x',y)} K(x',y) g(y)\, d\mu(y) \right| . \tag{3.10}$$

We split the first integral in (3.10) into the domains $r \leq \rho(x,y) < 2r$ and $2r \leq \rho(x,y)$. The integral over the first domain we estimate by (3.11) below. For the second domain, we observe with $\rho(x,x') \leq r$ and the triangle inequality that $r \leq \rho(x',y)$. We therefore combine on this domain with the corresponding part of the second integral in (3.10) and estimate that by (3.12) below. The remaining part of the second integral in (3.10) we estimate by (3.13). Overall, we have estimated (3.10) by

$$\int_{r \leq \rho(x,y) < 2r} |K(x,y)||g(y)|\, d\mu(y) \tag{3.11}$$

$$+ \left| \int_{2r \leq \rho(x,y)} (K(x,y) - K(x',y)) g(y)\, d\mu(y) \right| \tag{3.12}$$

$$+\int_{r\le\rho(x',y),r\le\rho(x,y)<2r}|K(x',y)||g(y)|\,d\mu(y)\,. \tag{3.13}$$

Using the bound on K in (1.7) and the doubling condition (1.4), we estimate (3.11) by

$$\int_{r\le\rho(x,y)<2r}\frac{2^{a^3}}{V(x,y)}|g(y)|\,d\mu(y)\ \le\ \frac{2^{a^3}}{\mu(B(x,r))}\int_{r\le\rho(x,y)<2r}|g(y)|\,d\mu(y) \tag{3.14}$$

$$\le\frac{2^{a^3}\cdot 2^a}{\mu(B(x,2r))}\int_{\rho(x,y)<2r}|g(y)|\,d\mu(y)\,. \tag{3.15}$$

Using the definition of Mg, we estimate (3.15) by

$$\le 2^{a^3+a}Mg(x)\,. \tag{3.16}$$

Similarly, in the domain of (3.13) we note by the triangle inequality and assumption on x' that $\rho(x',y)<3r$ and thus we estimate (3.13) by

$$\frac{2^{a^3}}{\mu(B(x',r))}\int_{\rho(x',y)<4r}|g(y)|\,d\mu(y)\le 2^{a^3+2a}Mg(x) \tag{3.17}$$

We turn to the remaining term. Using (3.1), we estimate (3.12) by

$$\int_{2r\le\rho(x,y)}\left(\frac{\rho(x,x')}{\rho(x,y)}\right)^{\frac{1}{a}}\frac{2^{a^3}}{V(x,y)}|g(y)|\,d\mu(y) \tag{3.18}$$

We decompose and estimate (3.12) with the triangle inequality by

$$\sum_{j=1}^{\infty}\int_{2^jr\le\rho(x,y)<2^{j+1}r}\left(\frac{\rho(x,x')}{\rho(x,y)}\right)^{\frac{1}{a}}\frac{2^{a^3}}{V(x,y)}|g(y)|\,d\mu(y) \tag{3.19}$$

$$\le\sum_{j=1}^{\infty}\left(2^{-j}\right)^{\frac{1}{a}}\int_{2^jr\le\rho(x,y)<2^{j+1}r}\frac{2^{a^3}}{\mu(B(x,2^jr))}|g(y)|\,d\mu(y) \tag{3.20}$$

$$\le\sum_{j=1}^{\infty}2^{-\frac{j}{a}}\frac{2^{a^3+a}}{\mu(B(x,2^{j+1}r))}\int_{\rho(x,y)<2^{j+1}r}|g(y)|\,d\mu(y) \tag{3.21}$$

$$\le 2^{a^3+a}\sum_{j=1}^{\infty}2^{-\frac{j}{a}}Mg(x)\,. \tag{3.22}$$

Using Lemma 3.5, we estimate (3.22) by

$$\le 2^{a^3+2a}Mg(x)\,. \tag{3.23}$$

Summing the estimates for (3.11), (3.12), and (3.13) proves the lemma. □

Lemma 3.7 (Cotlar control)**.** *Let $0 < r \leq R$ and $x \in X$. Let $g : X \to \mathbb{C}$ be a bounded measurable function supported on a set of finite measure. Then for all $x' \in X$ with $\rho(x, x') \leq \frac{R}{4}$ we have*

$$|T_R g(x)| \leq |T_r(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x')| + 2^{a^3+4a+1} Mg(x)\,. \tag{3.24}$$

Proof. Let x and x' be given with $\rho(x, x') \leq \frac{R}{4}$. By an application of Lemma 3.6, we estimate the left-hand-side of (3.24) by

$$|T_R(g)(x')| + 2^{a^3+2a+2} Mg(x)\,. \tag{3.25}$$

We have

$$T_R(g)(x') = \int_{R \leq \rho(x',y)} K(x', y)g(y)\, d\mu(y)\,. \tag{3.26}$$

On the domain $R \leq \rho(x', y)$, we have $\frac{R}{2} \leq \rho(x, y)$. Hence we may write for (3.26)

$$\begin{aligned} T_R(g)(x') &= \int_{R \leq \rho(x',y)} K(x', y)(g - g\mathbf{1}_{B(x,\frac{R}{2})})(y)\, d\mu(y) \\ &= T_R(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x')\,. \end{aligned} \tag{3.27}$$

Combining the estimate (3.25) with the identification (3.27), we obtain

$$|T_R g(x)| \leq |T_R(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x')| + 2^{a^3+2a+2} Mg(x)\,. \tag{3.28}$$

We have

$$\begin{aligned} &(T_r - T_R)(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x') \\ = &\int_{B(x',R)\setminus B(x',r)} K(x', y)(g - g\mathbf{1}_{B(x,\frac{R}{2})})(y)\, d\mu(y) \\ = &\int_{B(x',R)\setminus(B(x',r)\cup B(x,\frac{R}{2}))} K(x', y)g(y)\, d\mu(y) \end{aligned} \tag{3.29}$$

As $\frac{R}{2} \leq \rho(x, y)$ together with $\rho(x, x') \leq \frac{R}{4}$ implies $\frac{R}{4} \leq \rho(x', y)$, we can estimate the absolute value of (3.29) with (1.7) by

$$\begin{aligned} \leq &\frac{2^{a^3}}{\mu(B(x', \frac{R}{4}))} \int_{B(x,2R)\setminus B(x',\frac{R}{4})} |g(y)|\, d\mu(y) \\ \leq &\frac{2^{a^3+a}}{\mu(B(x', \frac{R}{2}))} \int_{B(x,2R)} |g(y)|\, d\mu(y) \\ \leq &2^{a^3+a} \frac{\mu(B(x, 2R))}{\mu(B(x, \frac{R}{4}))} Mg(x) \leq 2^{a^3+4a} Mg(x)\,. \end{aligned}$$

By the triangle inequality, (3.24) follows now from (3.28) and the estimate for (3.29). □

Lemma 3.8 (Cotlar sets)**.** *Assume that* (3.3) *holds. Let* $0 < r \leq R$ *and* $x \in X$*. Let* $g : X \to \mathbb{C}$ *be a bounded measurable function supported on a set of finite measure. Then the measure* $|F_1|$ *of the set* F_1 *of all* $x' \in B(x, \frac{R}{4})$ *such that*

$$|T_r g(x')| > 4M(T_r g)(x) \tag{3.30}$$

is less than or equal to $\mu(B(x, \frac{R}{4}))/4$*. Moreover, the measure* $|F_2|$ *of the set* F_2 *of all* $x' \in B(x, \frac{R}{4})$ *such that*

$$|T_r(g\mathbf{1}_{B(x,\frac{R}{2})})(x')| > 2^{a^3+20a+2} Mg(x) \tag{3.31}$$

is less than or equal to $\mu(B(x, \frac{R}{4}))/4$*.*

Proof. Let r, R, x and g be given. If $M(T_r g)(x) = 0$, then $T_r g$ is zero almost everywhere and the estimate on $|F_1|$ is trivial. Assume $M(T_r g)(x) > 0$. We have with (3.30)

$$M(T_r g)(x) \geq \frac{1}{\mu(B(x, \frac{R}{4}))} \int_{B(x,\frac{R}{4})} |T_r g(x')| \, dx' \tag{3.32}$$

$$\geq \frac{1}{\mu(B(x, \frac{R}{4}))} \int_{F_1} 4M(T_r g)(x) \, dx' \,. \tag{3.33}$$

Dividing by $M(T_r g)(x)$ gives

$$1 \geq \frac{4}{\mu(B(x, \frac{R}{4}))} |F_1| \,. \tag{3.34}$$

This gives the desired bound for the measure of F_1. We turn to the set F_2. Similarly as above we may assume $Mg(x) > 0$. The set F_2 is then estimated with Lemma 3.4 by

$$\frac{2^{a^3+19a}}{2^{a^3+20a+2} Mg(x)} \int |g\mathbf{1}_{B(x,\frac{R}{2})}|(y) \, d\mu(y) \tag{3.35}$$

$$\leq \frac{1}{2^{a+2} Mg(x)} \mu(B(x, \frac{R}{2})) Mg(x) \leq \frac{\mu(B(x, \frac{R}{4}))}{4} \,. \tag{3.36}$$

This gives the desired bound for the measure of F_2. □

Lemma 3.9 (Cotlar estimate)**.** *Assume that* (3.3) *holds. Let* $0 < r \leq R$ *and* $x \in X$*. Let* $g : X \to \mathbb{C}$ *be a bounded measurable function supported on a set of finite measure. Then*

$$|T_R g(x)| \leq 2^2 M(T_r g)(x) + 2^{a^3+20a+3} Mg(x) \,. \tag{3.37}$$

Proof. By Lemma 3.8, the set of all $x' \in B(x, \frac{R}{4})$ such that at least one of the conditions (3.30) and (3.31) is satisfied has measure less than or equal to $\mu(B(x, \frac{R}{4}))/2$ and hence is not all of $B(x, \frac{R}{4})$. Pick an $x' \in B(x, \frac{R}{4})$ such that both conditions are not satisfied. Applying Lemma 3.7 for this x' and using the triangle inequality estimates the left-hand side of (3.37) by

$$4M(T_r g)(x) + 2^{a^3+20a+2} Mg(x) + 2^{a^3+4a+1} Mg(x) \,. \tag{3.38}$$

This proves the lemma. □

Lemma 3.10 (simple nontangential operator)**.** *Assume that* (3.3) *holds. For every* $r > 0$ *and every bounded measurable function* g *supported on a set of finite measure we have*

$$\|T_*^r g\|_2 \le 2^{a^3+24a+6} \|g\|_2, \tag{3.39}$$

where

$$T_*^r g(x) := \sup_{r<R} \sup_{x' \in B(x,R)} |T_R(g)(x')| \,. \tag{3.40}$$

Proof. With Lemma 3.6 and the triangle inequality, we estimate for every $x \in X$

$$T_*^r g(x) \le 2^{a^3+2a+2} Mg(x) + \sup_{r<R} |T_R(g)(x)| \,. \tag{3.41}$$

Using further Lemma 3.9, we estimate

$$T_*^r g(x) \le 2^{a^3+2a+2} Mg(x) + 2^{a^3+20a+3} Mg(x) + 2^2 M(T_r g)(x) \,. \tag{3.42}$$

Taking the L^2 norm and using Theorem 3.1 with $a = 4$ and $p_2 = 2$ and $p_1 = 1$, we obtain

$$\|T_*^r g\|_2 \le 2^{a^3+20a+4} \|Mg\|_2 + 2^2 \|M(T_r g)\|_2 \tag{3.43}$$

$$\le 2^{a^3+24a+5} \|g\|_2 + 2^{4a+3} \|T_r g\|_2 \,. \tag{3.44}$$

Applying (3.3) gives

$$\|T_*^r g\|_2 \le 2^{a^3+24a+5} \|g\|_2 + 2^{a^3+4a+3} \|g\|_2 \,. \tag{3.45}$$

This shows (3.39) and completes the proof of the lemma. □

In order to pass from the one-sided truncation in T_r and T_*^r to the two-sided truncation in T_*, we show in the following two lemmas that the integral in (1.9) can be exchanged for an integral over the difference of two balls.

Lemma 3.11. *Let $f : X \to \mathbb{C}$ be a bounded measurable function supported on a set of finite measure. Let $x \in X$ and $R > 0$. Then, for all $\epsilon > 0$, there exists some $\delta > 0$ such that*

$$\left| \int_{R<\rho(x,y)<R+\delta} K(x,y) f(y) \, d\mu(y) \right| \le \epsilon \tag{3.46}$$

and

$$\left| \int_{R-\delta<\rho(x,y)<R} K(x,y) f(y) \, d\mu(y) \right| \le \epsilon \, . \tag{3.47}$$

Proof. We only prove the second inquality, the first one is analogous. Note that the integrand is bounded in $X \setminus B(x, \frac{R}{2})$. So for $0 < \delta \le \frac{R}{2}$,

$$\begin{aligned} &\left| \int_{R-\delta<\rho(x,y)<R} K(x,y) f(y) \, d\mu(y) \right| \\ \le & \frac{2^{a^3}}{\mu(B(x, \frac{R}{2}))} \sup_{y \in X} |f(x)| \cdot \mu(\{y \in X : R - \delta < \rho(x,y) < R\}). \end{aligned}$$

By continuity from above of μ, the right factor becomes arbitrarily small as $\delta \to 0$. Thus, for small enough δ, the whole expression is $\le \epsilon$. □

Lemma 3.12. *Let $f : X \to \mathbb{C}$ be a bounded measurable function supported on a set of finite measure. For all $x \in X$,*

$$T_* f(x) = \sup_{R_1 < R_2} \sup_{x' \in B(x,R_1)} \left| \int_{B(x',R_2) \setminus B(x',R_1)} K(x',y) f(y) \, \mathrm{d}\mu(y) \right| \tag{3.48}$$

Proof. We show two inequalities. Let $\epsilon > 0$. Let $R_1 < R_2$ and $x' \in B(x, R_1)$. Then for small enough $\delta > 0$,

$$\left| \int_{R_1<\rho(x',y)<R_2} K(x',y) f(y) \, \mathrm{d}\mu(y) \right| \tag{3.49}$$

$$\le \left| \int_{R_1<\rho(x',y)<R_1+\delta} K(x',y) f(y) \, \mathrm{d}\mu(y) \right| \tag{3.50}$$

$$+ \left| \int_{R_1+\delta\le\rho(x',y)<R_2} K(x',y) f(y) \, \mathrm{d}\mu(y) \right| . \tag{3.51}$$

By Lemma 3.11, we can choose δ such that (3.50) is bounded by ϵ. Without loss of generality, we can assume $R_1 + \delta < R_2$. Then (3.51) is bounded by the right hand side of (3.48) and we obtain

$$\le \epsilon + \sup_{R_1 < R_2} \sup_{x' \in B(x,R_1)} \left| \int_{B(x',R_2) \setminus B(x',R_1)} K(x',y) f(y) \, \mathrm{d}\mu(y) \right| .$$

The inequality still holds when taking the suprema over $R_1 < R_2$ and $\rho(x, x') < R_1$ in (3.49). Since $\epsilon > 0$ was arbitrary, this proves the first inequality.

The other direction is similar. Let $\epsilon > 0$. Let $R_1 < R_2$ and $x' \in B(x, R_1)$. Then for $\delta > 0$,

$$\left| \int_{B(x',R_2)\setminus B(x',R_1)} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| \tag{3.52}$$

$$\leq \left| \int_{R_1-\delta<\rho(x',y)<R_1} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| \tag{3.53}$$

$$+ \left| \int_{R_1-\delta<\rho(x',y)<R_2} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| . \tag{3.54}$$

By Lemma 3.11, we can choose δ such that (3.53) is bounded by ϵ. Without loss of generality, we can assume $\rho(x, x') < R_1 - \delta$. Then (3.54) is bounded by the left hand side of (3.48) and we obtain

$$\leq \epsilon + \sup_{R_1<R_2} \sup_{x'\in B(x,R_1)} \left| \int_{R_1<\rho(x',y)<R_2} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| .$$

The inequality still holds when taking the suprema over $R_1 < R_2$ and $\rho(x, x') < R_1$ in (3.49). Since $\epsilon > 0$ was arbitrary, this proves the second inequality. □

Proof of Lemma 3.3. Fix g as in the Lemma. Applying Lemma 3.10 with a sequence of r tending to 0 and using Lebesgue monotone convergence shows

$$\|T_*^0 g\|_2 \leq 2^{a^3+24a+6} \|g\|_2, \tag{3.55}$$

where

$$T_*^0 g(x) := \sup_{0<R} \sup_{x'\in B(x,R)} \left| \int_{X\setminus B(x',R)} K(x', y) g(y) \, d\mu(y) \right| . \tag{3.56}$$

We now write using Lemma 3.12 and the triangle inequality,

$$T_* g(x) \leq \sup_{0<R_1<R_2} \sup_{x'\in B(x,R_1)} \left| \int_{X\setminus B(x',R_1)} K(x', y) g(y) \, d\mu(y) \right|$$

$$+ \sup_{0<R_1<R_2} \sup_{x'\in B(x,R_1)} \left| \int_{X\setminus B(x',R_2)} K(x', y) g(y) \, d\mu(y) \right| .$$

Noting that the first integral does not depend on R_2 and estimating the second summand by the larger supremum over all $x' \in B(x, R_2)$, at which time the integral does not depend on R_1, we estimate further

$$\leq \sup_{0<R_1} \sup_{x' \in B(x,R_1)} \left| \int_{X \setminus B(x',R_1)} K(x',y) g(y) \, d\mu(y) \right|$$
$$+ \sup_{0<R_2} \sup_{x' \in B(x,R_2)} \left| \int_{X \setminus B(x',R_2)} K(x',y) g(y) \, d\mu(y) \right| .$$

Applying the triangle inequality on the left-hand side of (3.5) and applying (3.55) twice proves (3.5). This completes the proof of Lemma 3.3. □

3.1.2 Calderón-Zygmund Decomposition

Calderón-Zygmund decomposition is a tool to extend L^2 bounds to L^p bounds with $p < 2$ or to the so-called weak $(1, 1)$ type endpoint bound. It is classical and can be found in [Ste93].

The following lemma is Theorem 3.1(b) in [Ste93]. The proof uses Theorem 3.1.

Lemma 3.13 (Maximal theorem)**.** *Let $f : X \to \mathbb{C}$ be bounded, measurable, supported on a set of finite measure, and let $\alpha > 0$. Then*

$$\mu(\{x \in X : Mf(x) > \alpha\}) \leq \frac{2^{2a}}{\alpha} \int |f(y)| \, d\mu(y). \tag{3.57}$$

Proof. By definition, for each $x \in X$ with $Mf(x) > \alpha$, there exists a ball B_x such that $x \in B_x$ and

$$\alpha\mu(B_x) < \int_{B_x} |f(y)| \, d\mu(y). \tag{3.58}$$

Since $\{x \in X : Mf(x) > \alpha\}$ is open and μ is inner regular on open sets, it suffices to show that

$$\mu(E) \leq \frac{2^{2a}}{\alpha} \int |f(y)| \, d\mu(y)$$

for every compact $E \subset \{x \in X : Mf(x) > \alpha\}$. For such an E, by compactness, we can select a finite subcollection $\mathcal{B} \subset \{B_x : x \in E\}$ that covers E. By Theorem 3.1 applied to (3.58),

$$\alpha\mu(\bigcup \mathcal{B}) \leq 2^{2a} \int |f(y)| \, d\mu(y) \tag{3.59}$$

and hence

$$\mu(E) \leq \mu(\bigcup \mathcal{B}) \leq \frac{2^{2a}}{\alpha} \int |f(y)| \, d\mu(y).$$

□

Lemma 3.14 (Lebesgue differentiation)**.** *Let f be a bounded measurable function supported on a set of finite measure. Then for μ almost every x, we have*

$$\lim_{n\to\infty} \frac{1}{\mu(B_n)} \int_{B_n} f(y)\, dy = f(x),$$

where $\{B_n\}_{n\geq 1}$ is a sequence of balls with radii $r_n > 0$ such that $x \in B_n$ for each $n \geq 1$ and

$$\lim_{n\to\infty} r_n = 0\,.$$

Proof. This follows from the Lebesgue differentiation theorem, which is already formalized in Lean. □

Lemma 3.15 (Disjoint family countable)**.** *In a doubling metric measure space (X, ρ, μ, a), every disjoint family of balls $B_j = B(x_j, r_j)$, $j \in J$, is countable.*

Proof. Choose an arbitrary $x \in X$ as reference point. For $q, Q \in \mathbb{Q}_+$, let $J_{q,Q}$ denote the set of all $j \in J$ such that $B_j \subset B(x, Q)$ and $r_j \geq q$. It suffices to show that all the $J_{q,Q}$ are finite. Indeed, for all $j \in J_{q,Q}$,

$$\mu(B(x, Q)) \leq \mu(B(x_j, 2Q)) = \mu(B(x_j, \frac{2Q}{r_j} r_j)) \leq 2^{a \log_2 \lceil \frac{2Q}{r_j} \rceil} \mu(B_j).$$

Since the B_j are disjoint,

$$|J_{q,Q}| \mu(B(x, Q)) \leq 2^{a \log_2 \lceil \frac{2Q}{q} \rceil} \sum_{j \in J_{q,Q}} \mu(B_j) \leq 2^{a \log_2 \lceil \frac{2Q}{q} \rceil} \mu(B(x, Q)) \quad (3.60)$$

and hence $|J_{q,Q}| \leq 2^{a \log_2 \lceil \frac{2Q}{q} \rceil}$. □

The following lemma corresponds to Lemma 3.2 in [Ste93] with additional proof of the bounded intersection property taken from the proof of Proposition 7.1 .

Lemma 3.16 (Ball covering)**.** *Given an open set $O \neq X$, there exists a countable family of balls $B_j = B(x_j, r_j)$ such that*

$$B_j \cap B_{j'} = \emptyset \quad \textit{for } j \neq j' \quad (3.61)$$

and for $B_j^* := B(x_j, 3r_j)$,

$$\bigcup_j B_j^* = O \quad (3.62)$$

and for $B_j^{**} := B(x_j, 7r_j)$,

$$B_j^{**} \cap (X \setminus O) \neq \emptyset \quad \textit{for all } j \quad (3.63)$$

and we have the bounded intersection property that each $x \in O$ is contained in at most 2^{6a} of the B_j^*.

Proof. Define for $x \in O$,

$$\delta(x) := \sup\{\delta \in \mathbb{R} : B(x, \delta) \subset O\}. \tag{3.64}$$

Since O is open, and $O \neq X$, we have

$$0 < \delta(x) < \infty\,. \tag{3.65}$$

Using Zorn's Lemma, we select a maximal disjoint subfamily of $\{B(x, \frac{\delta(x)}{6}) : x \in O\}$. We obtain a (by Lemma 3.15 countable) family of balls $B_j = B(x_j, \frac{\delta(x_j)}{6}), j \in J$ such that (3.61). (3.63) and $\bigcup_j B_j^* \subset O$ are also immediate. For the other inclusion, first observe that for $x, y \in X$, if $B(x, \frac{\delta(x)}{6}) \cap B(y, \frac{\delta(y)}{6}) \neq \emptyset$, then

$$\delta(x) \leq \rho(x, y) + \delta(y) \leq (\frac{\delta(x)}{6} + \frac{\delta(y)}{6}) + \delta(y) = \frac{\delta(x)}{6} + \frac{7\delta(y)}{6},$$

so

$$\delta(x) \leq 2\delta(y). \tag{3.66}$$

Now let $z \in X$. By maximality, there exists some $j \in J$ with $B(z, \frac{\delta(z)}{6}) \cap B_j \neq \emptyset$. By (3.66),

$$\rho(z, x_j) < \frac{\delta(z)}{6} + \frac{\delta(x_j)}{6} \leq \frac{3\delta(x_j)}{6} = 3r_j$$

and thus $z \in B_j^*$.

We now turn to the bounded intersection property. Assume that for some $j_1, \dots, j_N$,

$$z \in \bigcap_{k=1}^{N} B_{j_k}^*. \tag{3.67}$$

Similarly as above, observe for $1 \leq k \leq N$,

$$\delta(z) \leq \rho(z, x_{j_k}) + \delta(x_{j_k}) \leq \frac{\delta(x_{j_k})}{2} + \delta(x_{j_k}) = \frac{3\delta(x_{j_k})}{2} \tag{3.68}$$

and

$$\delta(x_{j_k}) \leq \rho(x_{j_k}, z) + \delta(z) \leq \frac{\delta(x_{j_k})}{2} + \delta(z),$$

so

$$\delta(x_{j_k}) \leq 2\delta(z). \tag{3.69}$$

By (3.68) and (3.69), for all $1 \leq k \leq N$, $B(z, \frac{\delta(z)}{6}) \subset B(x_{j_k}, 5r_{j_k})$ and $B_{j_k} \subset B(z, \frac{8\delta(z)}{6})$. Using this and (3.61), we obtain

$$N\mu(B(z, \frac{\delta(z)}{6})) \leq \sum_{k=1}^{N} \mu(B(x_{j_k}, 5r_j)) \leq 2^{3a} \sum_{k=1}^{N} \mu(B_{j_k}) \tag{3.70}$$

$$= 2^{3a}\mu(\bigcup_{k=1}^{N} B_{j_k}) \leq 2^{3a}\mu(B(z, \frac{8\delta(z)}{6})) \leq 2^{6a}\mu(B(z, \frac{\delta(z)}{6})) \tag{3.71}$$

and conclude $N \le 2^{6a}$. □

Most of the next lemma and its proof is taken from Theorem 4.2 in [Ste93].

Lemma 3.17 (Calderon Zygmund decomposition)**.** *Let f be a bounded, measurable function supported on a set of finite measure and let $\alpha > \frac{1}{\mu(X)} \int |f| \, d\mu$. Then there exists a measurable function g, a countable family of balls B_j^* (where we allow $B_1^* = X$ in the special case that $\mu(X) < \infty$) such that each $x \in X$ is contained in at most 2^{6a} of the B_j^*, and a countable family of measurable functions $\{b_j\}_{j \in J}$ such that for all $x \in X$*

$$f(x) = g(x) + \sum_j b_j(x) \tag{3.72}$$

and such that the following holds. For almost every $x \in X$,

$$|g(x)| \le 2^{3a} \alpha \,. \tag{3.73}$$

We have

$$\int |g(y)| \, d\mu(y) \le \int |f(y)| \, d\mu(y). \tag{3.74}$$

For every j

$$\operatorname{supp} b_j \subset B_j^* \,. \tag{3.75}$$

For every j

$$\int_{B_j^*} b_j(x) \, d\mu(x) = 0, \tag{3.76}$$

and

$$\int_{B_j^*} |b_j(x)| \, d\mu(x) \le 2^{2a+1} \alpha \mu(B_j^*). \tag{3.77}$$

We have

$$\sum_j \mu(B_j^*) \le \frac{2^{4a}}{\alpha} \int |f(y)| \, d\mu(y) \tag{3.78}$$

and

$$\sum_j \int_{B_j^*} |b_j(y)| \, d\mu(y) \le 2 \int |f(y)| \, d\mu(y) \,. \tag{3.79}$$

Proof. Let $E_\alpha := \{x \in X : Mf(x) > \alpha\}$. Then E_α is open. Assume first that $E_\alpha \ne X$. We apply Lemma 3.16 with $O = E_\alpha$ to obtain the family $B_j, j \in J$,. Without loss of generality, we can assume $J = \mathbb{N}$. Define inductively

$$Q_j := B_j^* \setminus \left(\bigcup_{i<j} Q_i \cup \bigcup_{i>j} B_i \right) . \tag{3.80}$$

Then $B_j \subset Q_j \subset B_j^*$, the Q_j are pairwise disjoint and $\bigcup_j Q_j = E_\alpha$. Define

$$g(x) := \begin{cases} f(x), & x \in X \setminus E_\alpha, \\ \frac{1}{\mu(Q_j)} \int_{Q_j} f(y)\, d\mu(y), & x \in Q_j, \end{cases} \tag{3.81}$$

and, for each j,

$$b_j(x) := \mathbf{1}_{Q_j}(x) \left(f(x) - \frac{1}{\mu(Q_j)} \int_{Q_j} f(y)\, d\mu(y) \right). \tag{3.82}$$

Then (3.72), (3.75) and (3.76) are true by construction. For (3.73), we first do the case $x \in X \setminus E_\alpha$. By definition of Mf,

$$\frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y) \le \alpha \tag{3.83}$$

for every ball $B \subset X$ with $x \in B$. It follows by Lemma 3.14 that for almost every $x \in X \setminus E_\alpha$, $|f(x)| \le \alpha$. In the case $x \in E_\alpha$, there exists some $j \in J$ with $x \in Q_j$ and we have that

$$\frac{1}{\mu(B_j^{**})} \int_{B_j^{**}} |f(y)|\, d\mu(y) \le \alpha \tag{3.84}$$

because $B_j^{**} \cap (X \setminus E_\alpha) \neq \emptyset$. We get

$$|g(x)| \le \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)|\, d\mu(y) \le \frac{1}{\mu(B_j)} \int_{B_j^{**}} |f(y)|\, d\mu(y) \le 2^{3a}\alpha. \tag{3.85}$$

To prove (3.74), we estimate

$$\begin{aligned} \int |g(z)|\, d\mu(z) &\le \int_{X \setminus E_\alpha} |f(z)|\, d\mu(z) + \sum_j \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)|\, d\mu(y)\, d\mu(z) \\ &= \int |f(z)|\, d\mu(z). \end{aligned}$$

Using the triangle inequality, we have that

$$\int_{B_j^*} |b_j(y)|\, dy \le \int_{Q_j} |f(y)|\, d\mu(y) + \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)|\, d\mu(x)\, d\mu(y) \tag{3.86}$$

$$= 2 \int_{Q_j} |f(y)|\, dy. \tag{3.87}$$

With (3.84), we estimate further

$$\le 2 \int_{B_j^{**}} |f(y)|\, dy \le 2\mu(B_j^{**})\alpha \le 2^{2a+1}\alpha\mu(B_j^*) \tag{3.88}$$

to obtain (3.77). Further, summing up (3.86) in j yields (3.79). At last, we estimate with Lemma 3.13

$$\sum_j \mu(B_j^*) \le 2^{2a} \sum_j \mu(B_j) \le 2^{2a}\mu(E_\alpha) \le \frac{2^{4a}}{\alpha} \int |f(y)|\, d\mu(y), \tag{3.89}$$

proving (3.78).

Assume now that $E_\alpha = X$. It follows from Lemma 3.13 that then $\mu(X) < \infty$. Define

$$g := \frac{1}{\mu(X)} \int |f(y)|\, d\mu(y)$$

and

$$b_1 := f - g.$$

Then $f = g + b_1$ and $\operatorname{supp} b_1 \subset B_1^* := X$ and (3.72), (3.74), (3.75), (3.76) all hold immediately. By assumption, $\alpha > \frac{1}{\mu(X)} \int |f|\, d\mu = g$, so (3.73) holds. We also have, using the definitions and the same assumption,

$$\int |b_1(y)|\, d\mu(y) \le 2 \int |f(y)|\, d\mu(y) \le 2\alpha\mu(X), \tag{3.90}$$

which verifies both (3.79) and (3.77). Finally, by Lemma 3.13,

$$\mu(X) \le \frac{2^{2a}}{\alpha} \int |f(y)|\, d\mu(y),$$

which shows (3.78). □

We use Lemma 3.17 to prove Lemma 3.4. For the remainder of this section, let $f : X \to \mathbb{C}$, $r > 0$ and $\alpha > 0$ as in the lemma. We define the constant

$$c := 2^{-a^3-12a-4} \tag{3.91}$$

and $\alpha' := c\alpha$. If $\alpha' \le \frac{1}{\mu(X)} \int |f|\, d\mu$, then we directly have

$$\mu\left(\{x \in X : |T_r f(x)| > \alpha\}\right) \le \mu(X) \le \frac{1}{\alpha'} \int |f(y)|\, d\mu(y)$$

$$\le \frac{2^{a^3+19a}}{\alpha} \int |f(y)|\, d\mu(y),$$

which proves (3.9). So assume from now on that $\alpha' > \frac{1}{\mu(X)} \int |f|\, d\mu$. Using Lemma 3.17 for f and α', we obtain the decomposition

$$f = g + b = g + \sum_j b_j$$

such that the properties (3.72)-(3.79) are satisfied (with α' replacing α). We rename B_j^* to B_j and let

$$B_j = B(x_j, r_j). \tag{3.92}$$

Define

$$B_j' := B(x_j, 2r_j). \tag{3.93}$$

(In the special case $B_j = X$, we define $B_j' := X$.) Then B_j' is a ball with the same center as B_j but with

$$\mu(B_j') \le 2^a \mu(B_j). \tag{3.94}$$

Let

$$\Omega := \bigcup_j B_j'. \tag{3.95}$$

We deal with $T_r g$ and $T_r b$ separately in the following lemmas.

Lemma 3.18 (Estimate good)**.**

$$\mu\left(\{x \in X : |T_r g(x)| > \alpha/2\}\right) \le \frac{2^{2a^3+3a+2}c}{\alpha} \int |f(y)|\, d\mu(y).$$

Proof. We estimate using monotonicity of the integral

$$\mu\left(\{x \in X : |T_r g(x)| > \alpha/2\}\right) \le \frac{4}{\alpha^2} \int |T_r g(y)|^2\, d\mu(y).$$

Using (3.8) followed by (3.73) and (3.74), we estimate the right hand side above by

$$\le \frac{4 \cdot 2^{2a^3}}{\alpha^2} \int |g(y)|^2\, d\mu(y) \le \frac{2^{2a^3+3a+2}c}{\alpha} \int |g(y)|\, dy$$

$$\le \frac{2^{2a^3+3a+2}c}{\alpha} \int |f(y)|\, d\mu(y). \tag{3.96}$$

□

Lemma 3.19. *Let $x \in X \setminus \Omega$. Then*

$$|T_r b(x)| \le 3F(x) + \alpha/8,$$

where

$$F(x) := 2^{a^3+2a+1} c\alpha \sum_{j \in J} \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{\mu(B_j)}{V(x, x_j)}.$$

Proof. We decompose the index set J into the following disjoint sets:

$$\mathcal{J}_1(x) := \{j\ : r + r_j \le \rho(x, x_j)\},$$
$$\mathcal{J}_2(x) := \{j\ : r - r_j \le \rho(x, x_j) < r + r_j\},$$
$$\mathcal{J}_3(x) := \{j\ : \rho(x, x_j) < r - r_j\}.$$

Then

$$|T_r b(x)| \le \sum_{j\in\mathcal{J}_1(x)} |T_r b_j(x)| \tag{3.97}$$
$$+ \sum_{j\in\mathcal{J}_2(x)} |T_r b_j(x)| \tag{3.98}$$
$$+ \sum_{j\in\mathcal{J}_3(x)} |T_r b_j(x)|. \tag{3.99}$$

For all $j \in \mathcal{J}_3(x)$, $\operatorname{supp} b_j \subset B_j \subset B(x, r)$, and thus $T_r b_j(x) = 0$, so (3.99) $= 0$.

Next, for $j \in \mathcal{J}_1(x)$, $\operatorname{supp} b_j \subset B_j \subset X \setminus B(x, r)$, and we have

$$T_r b_j(x) = \int_{X\setminus B(x,r)} K(x, y) b_j(y)\, d\mu(y) = \int_{B_j} K(x, y) b_j(y)\, d\mu(y)\,.$$

Using (3.76), the above is equal to

$$\int_{B_j} (K(x, y) - K(x, x_j)) b_j(y)\, d\mu(y)\,.$$

Since $x \in X \setminus \Omega$, we have for each $y \in B_j$ that

$$\rho(x, x_j) \ge 2r_j > 2\rho(x_j, y), \tag{3.100}$$

so we can apply (1.8) to estimate

$$(3.97) \le \sum_{j\in\mathcal{J}_1(x)} \int_{B_j} \left(\frac{\rho(x_j, y)}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x, x_j)} |b_j(y)|\, d\mu(y)$$

$$\le 2^{a^3} \sum_j \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{1}{V(x, x_j)} \int_{B_j} |b_j(y)|\, dy \tag{3.101}$$

and by (3.77),

$$\le 2^{a^3+2a+1} c\alpha \sum_j \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{\mu(B_j)}{V(x, x_j)} = F(x). \tag{3.102}$$

Next, we estimate (3.98). For each $j \in \mathcal{J}_2(x)$, set

$$d_j := \frac{1}{\mu(B_j)} \int_{B_j} \mathbf{1}_{X\setminus B(x,r)}(y) b_j(y)\, dy.$$

Then by (3.77)

$$|d_j| \le 2^{2a+1} c\alpha. \tag{3.103}$$

For each $j \in \mathcal{J}_2(x)$, we have

$$T_r b_j(x) = \int_{B_j} K(x,y)(\mathbf{1}_{X\setminus B(x,r)}(y) b_j(y) - d_j)\, dy + \int_{B_j} d_j K(x,y)\, dy$$

$$= \int_{B_j} (K(x,y) - K(x,x_j))(\mathbf{1}_{X\setminus B(x,r)}(y) b_j(y) - d_j)\, dy + \int_{B_j} d_j K(x,y)\, dy.$$

Thus, using the triangle inequality, the equation above and (3.103), we obtain

$$|T_r b_j(x)| \le$$

$$\int_{B_j} |K(x,y) - K(x,x_j)| \left(|b_j(y)| + 2^{2a+1} c\alpha\right)\, dy + 2^{2a+1} c\alpha \int_{B_j} |K(x,y)|\, dy. \tag{3.104}$$

By (3.100), we can apply (1.8) and arguing as in (3.102), we get that

$$(3.98) \le 2F(x) + 2^{2a+1} c\alpha \sum_{j \in \mathcal{J}_2(x)} \int_{B_j} |K(x,y)|\, d\mu(y), \tag{3.105}$$

with F as in (3.102). Define

$$A := \bigcup_{j \in \mathcal{J}_2(x)} B_j.$$

We claim that

$$A \subset B(x, 3r) \setminus B(x, \frac{r}{3}). \tag{3.106}$$

Indeed, for each $j \in \mathcal{J}_2(x)$ and $y \in B_j$, using again (3.100),

$$\rho(x,x_j) < r + r_j \le r + \frac{1}{2}\rho(x,x_j) \implies \rho(x,x_j) < 2r$$

and hence

$$\rho(x,y) \le \rho(x,x_j) + \rho(x_j,y) < 2r + r_j \le 2r + \frac{1}{2}\rho(x,x_j) < 3r.$$

For the lower bound, we observe

$$\rho(x,x_j) \ge r - r_j \ge r - \frac{1}{2}\rho(x,x_j) \implies \rho(x,x_j) \ge \frac{2}{3} r,$$

and conclude

$$\rho(x,y) \geq \rho(x,x_j) - \rho(y,x_j) \geq \rho(x,x_j) - r_j \geq \rho(x,x_j) - \frac{1}{2}\rho(x,x_j) \geq \frac{1}{3}r.$$

Using the bounded intersection property of the B_j, (3.106) and (1.7), we get

$$\sum_{j\in\mathcal{J}_2(x)} \int_{B_j} |K(x,y)|\, d\mu(y) \leq 2^{6a} \int_A |K(x,y)|\, d\mu(y) \tag{3.107}$$

$$\leq 2^{6a} \int_{B(x,3r)\setminus B(x,\frac{r}{3})} |K(x,y)|\, d\mu(y) \tag{3.108}$$

$$\leq 2^{6a} \int_{B(x,3r)\setminus B(x,\frac{r}{3})} \frac{2^{a^3}}{V(x,y)}\, d\mu(y) \tag{3.109}$$

$$\leq 2^{a^3+6a} \int_{B(x,3r)\setminus B(x,\frac{r}{3})} \frac{1}{\mu(B(x,\frac{r}{3}))}\, d\mu(y) \tag{3.110}$$

$$\leq 2^{a^3+6a} \frac{\mu(B(x,3r))}{\mu(B(x,\frac{r}{3}))} \tag{3.111}$$

$$\leq 2^{a^3+10a}. \tag{3.112}$$

Combining the estimates (3.102) for (3.97), (3.105) for (3.98), and (3.112), we get

$$|T_r b(x)| \leq 3F(x) + 2^{a^3+12a+1} c\alpha.$$

By the definition (3.91) of c, this equals

$$3F(x) + \alpha/8.$$

□

Lemma 3.20. *For F as defined in Lemma 3.19, we have*

$$\mu(\{x \in X \setminus \Omega : F(x) > \alpha/8\}) \leq \frac{2^{a^3+9a+4}}{\alpha} \int |f(y)|\, d\mu(y)\,. \tag{3.113}$$

Proof. We estimate

$$\mu(\{x \in X \setminus \Omega : F(x) > \alpha/8\}) \tag{3.114}$$

$$\leq \frac{8}{\alpha} \int_{X\setminus\Omega} F(x)\, d\mu(x) \tag{3.115}$$

$$\leq \frac{8}{\alpha} \int_{X\setminus\Omega} 2^{a^3+2a+1} c\alpha \sum_j \left(\frac{r_j}{\rho(x,x_j)}\right)^{\frac{1}{a}} \frac{\mu(B_j)}{V(x,x_j)}\, d\mu(x) \tag{3.116}$$

$$\leq 2^{a^3+2a+4} c \sum_j \mu(B_j) \int_{X\setminus B_j'} \left(\frac{r_j}{\rho(x,x_j)}\right)^{\frac{1}{a}} \frac{1}{V(x,x_j)}\, d\mu(x) \tag{3.117}$$

Using

$$V(x,x_j) = \mu(B(x,\rho(x,x_j))) \geq 2^{-a}\mu(B(x,2\rho(x,x_j)))$$
$$\geq 2^{-a}\mu(B(x_j,\rho(x_j,x))),$$

we have for all $j \in J$,

$$\begin{aligned}
&\int_{X\setminus B_j'} \left(\frac{r_j}{\rho(x,x_j)}\right)^{\frac{1}{a}} \frac{1}{V(x,x_j)}\, d\mu(x) \\
\leq& 2^a \int_{X\setminus B_j'} \left(\frac{r_j}{\rho(x,x_j)}\right)^{\frac{1}{a}} \frac{1}{\mu(B(x_j,\rho(x_j,x)))}\, d\mu(x) \\
\leq& 2^a \sum_{n=1}^{\infty} \int_{B(x_j,2^{n+1}r_j)\setminus B(x_j,2^n r_j)} \left(\frac{r_j}{2^n r_j}\right)^{\frac{1}{a}} \frac{1}{\mu(B(x_j,2^n r_j))}\, d\mu(x) \\
\leq& 2^a \sum_{n=1}^{\infty} 2^{-\frac{n}{a}} \frac{\mu(B(x_j,2^{n+1}r_j))}{\mu(B(x_j,2^n r_j))} \\
\leq& 2^{3a},
\end{aligned}$$

where we used Lemma 3.5 in the last step. Plugging this into (3.117) and using (3.78), we conclude that

$$\mu(\{x \in X \setminus \Omega : F(x) > \alpha/8\}) \leq \frac{2^{a^3+9a+4}}{\alpha} \int |f(y)|\, d\mu(y)\,.$$

□

Lemma 3.21 (Estimate bad)**.** *We have*

$$\mu\left(\{x \in X : |T_r b(x)| > \alpha/2\}\right) \leq \frac{\frac{2^{5a}}{c} + 2^{a^3+9a+4}}{\alpha} \int |f(y)|\, d\mu(y)\,.$$

Proof. We estimate

$$\mu\left(\{x \in X : |T_r b(x)| > \alpha/2\}\right)$$
$$\leq \mu(\Omega) + \mu\left(\{x \in X \setminus \Omega : |T_r b(x)| > \alpha/2\}\right)\,. \tag{3.118}$$

Using (3.94) and (3.78), we conclude that

$$\mu(\Omega) \leq \sum_j \mu(B_j') \leq 2^a \sum_j \mu(B_j) \leq \frac{2^{5a}}{c\alpha} \int |f(y)|\, d\mu(y)\,. \tag{3.119}$$

It follows from Lemma 3.19 and the triangle inequality that

$$\mu(\{x \in X \setminus \Omega : |T_r b(x)| > \alpha/2\}) \leq \mu(\{x \in X \setminus \Omega : F(x) > \alpha/8\})\,. \tag{3.120}$$

The claim now follows from (3.118), (3.120) and (3.113). □

Proof of Lemma 3.4. It follows by the triangle inequality and subadditivity of μ that

$$\mu\left(\{x \in X : |T_r f(x)| > \alpha\}\right)$$
$$\leq \mu\left(\{x \in X : |T_r g(x)| > \alpha/2\}\right) + \mu\left(\{x \in X : |T_r b(x)| > \alpha/2\}\right).$$

Using Lemma 3.18, Lemma 3.21 and the definition (3.91) of c, we get

$$\begin{aligned}
&\leq \frac{2^{2a^3+3a+2}c + \frac{2^{5a}}{c} + 2^{a^3+9a+4}}{\alpha} \int |f(y)|\, d\mu(y) \\
&= \frac{2^{a^3-9a-2} + 2^{a^3+17a+4} + 2^{a^3+9a+4}}{\alpha} \int |f(y)|\, d\mu(y) \\
&\leq \frac{2^{a^3+19a}}{\alpha} \int |f(y)|\, d\mu(y).
\end{aligned}$$

This proves (3.9). □

3.2 Proof of The Classical Carleson Theorem

The convergence of partial Fourier sums is proved in Section 3.2.1 in two steps. In the first step, we establish convergence on a suitable dense subclass of functions. We choose smooth functions as subclass, the convergence is stated in Lemma 3.23 and proved in Section 3.2.2. In the second step, one controls the relevant error of approximating a general function by a function in the subclass. This is stated in Lemma 3.24 and proved in Section 3.2.6. The proof relies on a bound on the real Carleson maximal operator stated in Lemma 3.25 and proved in Section 3.2.7, which involves showing that the real line fits into the setting of Section 1.2. This latter proof refers to the two-sided variant of the Carleson Theorem 3.2. Two assumptions in Theorem 1.2 require more work. The boundedness of the operator T_r defined in (3.2) is established in 3.26. This lemma is proved in Section 3.2.3. The cancellative property is verified by Lemma 3.27, which is proved in Section 3.2.4. Several further auxiliary lemmas are stated and proved in Section 3.2.1, the proof of one of these auxiliary lemmas, Lemma 3.31, is done in Section 3.2.5.

All subsections past Section 3.2.1 are mutually independent.

3.2.1 The classical Carleson theorem

Let a uniformly continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ and $\epsilon > 0$ be given. Let

$$C_{a,q} := \frac{2^{452a^3}}{(q-1)^6} \tag{3.121}$$

denote the constant from Theorem 3.2. Define

$$\epsilon' := \frac{\epsilon}{4C_\epsilon}, \tag{3.122}$$

where

$$C_\epsilon = \left(\frac{8}{\pi\epsilon}\right)^{\frac{1}{2}} C_{4,2} + \pi \,.$$

Since f is continuous and periodic, f is uniformly continuous. Thus, there is a $0 < \delta < \pi$ such that for all $x, x' \in \mathbb{R}$ with $|x - x'| \leq \delta$ we have

$$|f(x) - f(x')| \leq \epsilon' \,. \tag{3.123}$$

Define

$$f_0 := f * \phi_\delta, \tag{3.124}$$

where ϕ_δ is a nonnegative smooth bump function with $\text{supp}(\phi_\delta) \subset (-\delta, \delta)$ and $\int_{\mathbb{R}} \phi_\delta(x)\, dx = 1$.

Lemma 3.22 (smooth approximation)**.** *The function f_0 is 2π-periodic. The function f_0 is smooth (and therefore measurable). The function f_0 satisfies for all $x \in \mathbb{R}$:*

$$|f(x) - f_0(x)| \leq \epsilon' \,, \tag{3.125}$$

Proof. Periodicity follows dirictly from the definitions. The other properties are part of the Lean library. □

We prove in Section 3.2.2:

Lemma 3.23 (convergence for smooth)**.** *There exists some $N_0 \in \mathbb{N}$ such that for all $N > N_0$ and $x \in [0, 2\pi]$ we have*

$$|S_N f_0(x) - f_0(x)| \leq \frac{\epsilon}{4} \,. \tag{3.126}$$

We prove in Section 3.2.6:

Lemma 3.24 (control approximation effect)**.** *There is a set $E \subset \mathbb{R}$ with Lebesgue measure $|E| \leq \epsilon$ such that for all*

$$x \in [0, 2\pi) \setminus E \tag{3.127}$$

we have

$$\sup_{N \geq 0} |S_N f(x) - S_N f_0(x)| \leq \frac{\epsilon}{4} \,. \tag{3.128}$$

We are now ready to prove classical Carleson:

Proof of Theorem 1.1. Let N_0 be as in Lemma 3.23. For every

$$x \in [0, 2\pi) \setminus E \,, \tag{3.129}$$

and every $N > N_0$ we have by the triangle inequality

$$|f(x) - S_N f(x)|$$

$$\leq |f(x) - f_0(x)| + |f_0(x) - S_N f_0(x)| + |S_N f_0(x) - S_N f(x)| \,. \tag{3.130}$$

Using Lemmas 3.22 to 3.24, we estimate (3.130) by

$$\leq \epsilon' + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \epsilon \,. \tag{3.131}$$

This shows (1.3) for the given E and N_0. $\square$

Let $\kappa : \mathbb{R} \to \mathbb{C}$ be the function defined by $\kappa(0) = 0$ and for $0 < |x| < 1$

$$\kappa(x) = \frac{1 - |x|}{1 - e^{ix}} \tag{3.132}$$

and for $|x| \geq 1$,

$$\kappa(x) = 0 \,. \tag{3.133}$$

Note that this function is continuous at every point x with $|x| > 0$.

The proof of Lemma 3.24 will use the following Lemma 3.25, which itself is proven in Section 3.2.7 as an application of Theorem 1.2.

Lemma 3.25 (real Carleson)**.** *Let F, G be Borel subsets of $\mathbb{R}$ with finite measure. Let f be a bounded measurable function on $\mathbb{R}$ with $|f| \leq \mathbf{1}_F$. Then*

$$\left| \int_G Tf(x)\, dx \right| \leq C_{4,2} |F|^{\frac{1}{2}} |G|^{\frac{1}{2}} \,, \tag{3.134}$$

where

$$Tf(x) = \sup_{n \in \mathbb{Z}} \sup_{r > 0} \left| \int_{r < |x-y| < 1} f(y) \kappa(x - y) e^{iny} \, dy \right| \,. \tag{3.135}$$

One of the main assumptions of Theorem 3.2, concerning the operator T_r defined in (3.2), is verified by the following lemma, which is proved in Section 3.2.3.

Lemma 3.26 (Hilbert strong 2 2)**.** *Let $0 < r < 1$. Let f be a bounded, measurable function on $\mathbb{R}$ with bounded support. Then*

$$\|H_r f\|_2 \leq 2^{13} \|f\|_2, \tag{3.136}$$

where

$$H_r f(x) := T_r f(x) = \int_{r \leq \rho(x,y)} \kappa(x - y) f(y) \, dy \tag{3.137}$$

The next lemma will be used to verify that the collection Θ of modulation functions in our application of Theorem 1.2 satisfies the condition of being cancellative. It is proved in Section 3.2.4.

Lemma 3.27 (van der Corput)**.** *Let $\alpha \le \beta$ be real numbers. Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable function and assume*

$$\|g\|_{Lip(\alpha,\beta)} := \sup_{\alpha \le x \le \beta} |g(x)| + \frac{|\beta - \alpha|}{2} \sup_{\alpha \le x < y \le \beta} \frac{|g(y) - g(x)|}{|y - x|} < \infty\,. \tag{3.138}$$

Then for any $\alpha \le \beta$ and $n \in \mathbb{Z}$ we have

$$\int_\alpha^\beta g(x) e^{inx}\, dx \le 2\pi |\beta - \alpha| \|g\|_{Lip(\alpha,\beta)} (1 + |n||\beta - \alpha|)^{-1}\,. \tag{3.139}$$

We close this section with six lemmas that are used across the following subsections.

Lemma 3.28 (mean zero oscillation)**.** *Let $n \in \mathbb{Z}$ with $n \neq 0$, then*

$$\int_0^{2\pi} e^{inx}\, dx = 0\,. \tag{3.140}$$

Proof. We have

$$\int_0^{2\pi} e^{inx}\, dx = \left[\frac{1}{in} e^{inx}\right]_0^{2\pi} = \frac{1}{in}(e^{2\pi in} - e^{2\pi i0}) = \frac{1}{in}(1 - 1) = 0\,. \qquad \square$$

Lemma 3.29 (Dirichlet kernel)**.** *We have for every 2π-periodic bounded measurable f and every $N \ge 0$*

$$S_N f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) K_N(x - y)\, dy \tag{3.141}$$

where K_N is the 2π-periodic continuous function of $\mathbb{R}$ given by

$$\sum_{n=-N}^{N} e^{inx'}\,. \tag{3.142}$$

We have for $e^{ix'} \neq 1$ that

$$K_N(x') = \frac{e^{iNx'}}{1 - e^{-ix'}} + \frac{e^{-iNx'}}{1 - e^{ix'}}\,. \tag{3.143}$$

Proof. We have by definitions and interchanging sum and integral

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}_n e^{inx}$$

$$= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{in(x-y)}\, dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(y) \sum_{n=-N}^{N} e^{in(x-y)}\, dy\,. \tag{3.144}$$

This proves the first statement of the lemma. By a telescoping sum, we have for every $x' \in \mathbb{R}$

$$\left(e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}\right) \sum_{n=-N}^{N} e^{inx'} = e^{(N+\frac{1}{2})ix'} - e^{-(N+\frac{1}{2})ix'}\,. \tag{3.145}$$

If $e^{ix'} \neq 1$, the first factor on the left-hand side is not 0 and we may divide by this factor to obtain

$$\sum_{n=-N}^{N} e^{inx'} = \frac{e^{i(N+\frac{1}{2})x'}}{e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}} - \frac{e^{-i(N+\frac{1}{2})x'}}{e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}} = \frac{e^{iNx'}}{1-e^{-ix'}} + \frac{e^{-iNx'}}{1-e^{ix'}}\,. \tag{3.146}$$

This proves the second part of the lemma. □

Lemma 3.30 (lower secant bound)**.** *Let $\eta > 0$ and $-2\pi + \eta \le x \le 2\pi - \eta$ with $|x| \ge \eta$. Then*

$$|1 - e^{ix}| \ge \frac{2}{\pi}\eta \tag{3.147}$$

Proof. We have

$$|1 - e^{ix}| = \sqrt{(1-\cos(x))^2 + \sin^2(x)} \ge |\sin(x)|\,.$$

If $0 \le x \le \frac{\pi}{2}$, then we have from concavity of sin on $[0, \pi]$ and $\sin(0) = 0$ and $\sin(\frac{\pi}{2}) = 1$

$$|\sin(x)| \ge \frac{2}{\pi}x \ge \frac{2}{\pi}\eta\,.$$

When $x \in \frac{m\pi}{2} + [0, \frac{\pi}{2}]$ for $m \in \{-4, -3, -2, -1, 1, 2, 3\}$ one can argue similarly. □

The following lemma will be proved in Section 3.2.5.

Lemma 3.31 (spectral projection bound)**.** *Let f be a bounded 2π-periodic measurable function. Then, for all $N \ge 0$*

$$\|S_N f\|_{L^2[-\pi,\pi]} \le \|f\|_{L^2[-\pi,\pi]}. \tag{3.148}$$

Lemma 3.32 (Hilbert kernel bound)**.** *For $x, y \in \mathbb{R}$ with $x \neq y$ we have*

$$|\kappa(x-y)| \le 2^2(2|x-y|)^{-1}\,. \tag{3.149}$$

Proof. Fix $x \neq y$. If $\kappa(x-y)$ is zero, then (3.149) is evident. Assume $\kappa(x-y)$ is not zero, then $0 < |x-y| < 1$. We have

$$|\kappa(x-y)| = \left|\frac{1-|x-y|}{1-e^{i(x-y)}}\right| . \tag{3.150}$$

We estimate with Lemma 3.30

$$|\kappa(x-y)| \leq \frac{1}{|1-e^{i(x-y)}|} \leq \frac{2}{|x-y|} . \tag{3.151}$$

This proves (3.149) in the given case and completes the proof of the lemma. □

Lemma 3.33 (Hilbert kernel regularity)**.** *For $x, y, y' \in \mathbb{R}$ with $x \neq y, y'$ and*

$$2|y-y'| \leq |x-y| , \tag{3.152}$$

we have

$$|\kappa(x-y) - \kappa(x-y')| \leq 2^8 \frac{1}{|x-y|} \frac{|y-y'|}{|x-y|} . \tag{3.153}$$

Proof. Upon replacing y by $y-x$ and y' by $y'-x$ on the left-hand side of (3.152), we can assume that $x=0$. Then the assumption (3.152) implies that y and y' have the same sign. Since $\kappa(y) = \bar{\kappa}(-y)$ we can assume that they are both positive. Then it follows from (3.152) that

$$\frac{y}{2} \leq y' .$$

We distinguish four cases. If $y, y' \leq 1$, then we have

$$|\kappa(-y) - \kappa(-y')| = \left|\frac{1-y}{1-e^{-iy}} - \frac{1-y'}{1-e^{-iy'}}\right|$$

and by the fundamental theorem of calculus

$$= \left|\int_{y'}^{y} \frac{-1+e^{-it}+i(1-t)e^{it}}{(1-e^{-it})^2}\, dt\right| .$$

Using $y' \geq \frac{y}{2}$ and Lemma 3.30, we bound this by

$$\leq |y-y'| \sup_{\frac{y}{2} \leq t \leq 1} \frac{3}{|1-e^{-it}|^2} \leq 3|y-y'|(2\frac{2}{y})^2 \leq 2^6 \frac{|y-y'|}{|y|^2} .$$

If $y \leq 1$ and $y' > 1$, then $\kappa(-y') = 0$ and we have from the first case

$$|\kappa(-y) - \kappa(-y')| = |\kappa(-y) - \kappa(-1)| \leq 2^6 \frac{|y-1|}{|y|^2} \leq 2^6 \frac{|y-y'|}{|y|^2} .$$

Similarly, if $y > 1$ and $y' \leq 1$, then $\kappa(-y) = 0$ and we have from the first case

$$|\kappa(-y) - \kappa(-y')| = |\kappa(-y') - \kappa(-1)| \leq 2^6 \frac{|y' - 1|}{|y'|^2} \leq 2^6 \frac{|y - y'|}{|y'|^2}.$$

Using again $y' \geq \frac{y}{2}$, we bound this by

$$\leq 2^6 \frac{|y - y'|}{|y/2|^2} = 2^8 \frac{|y - y'|}{|y|^2}$$

Finally, if $y, y' > 1$ then

$$|\kappa(-y) - \kappa(-y')| = 0 \leq 2^8 \frac{|y - y'|}{|y|^2}.$$

□

3.2.2 Smooth functions.

Lemma 3.34. *Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and differentiable, and let $n \in \mathbb{Z} \setminus \{0\}$. Then*

$$\widehat{f}_n = \frac{1}{in} \widehat{f'}_n. \tag{3.154}$$

Proof. This is part of the Lean library. □

Lemma 3.35. *Let $f : \mathbb{R} \to \mathbb{C}$ such that*

$$\sum_{n \in \mathbb{Z}} |\widehat{f}_n| < \infty. \tag{3.155}$$

Then

$$\sup_{x \in [0, 2\pi]} |f(x) - S_N f(x)| \to 0 \tag{3.156}$$

as $N \to \infty$.

Proof. This is part of the Lean library. □

Lemma 3.36. *Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and twice continuously differentiable. Then*

$$\sup_{x \in [0, 2\pi]} |f(x) - S_N f(x)| \to 0 \tag{3.157}$$

as $N \to \infty$.

Proof. By Lemma 3.35, it suffices to show that the Fourier coefficients $\widehat{f}_n$ are summable. Applying Lemma 3.34 twice and using the fact that f'' is continuous and thus bounded on $[0, 2\pi]$, we compute

$$\sum_{n \in \mathbb{Z}} |\widehat{f}_n| = |\widehat{f}_0| + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} |\widehat{f''}_n| \leq |\widehat{f}_0| + \left(\sup_{x \in [0, 2\pi]} |f(x)| \right) \cdot \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} < \infty.$$

□

Proof. Lemma 3.23 now follows directly from the previous Lemma 3.36. □

3.2.3 The truncated Hilbert transform

No changes have been made here compared to 10.4 of [Bec+24].

3.2.4 The proof of the van der Corput Lemma

Proof of Lemma 3.27. Let g be a Lipschitz continuous function as in the lemma. Assume first that $n = 0$. Then

$$\int_\alpha^\beta g(x)\,\mathrm{d}x \le |\beta-\alpha| \sup_{\alpha\le x\le\beta} |g(x)| \le |\beta-\alpha| \|g\|_{Lip(\alpha,\beta)}(1+|n||\beta-\alpha|)^{-1}$$

Assume now $n \neq 0$. Without loss of generality, we may assume $n > 0$. We distinguish two cases. If $\beta - \alpha < \frac{\pi}{n}$, we have by the triangle inequality

$$\left|\int_\alpha^\beta g(x)e^{inx}\,\mathrm{d}x\right| \le |\beta-\alpha| \sup_{x\in[\alpha,\beta]} |g(x)|$$

$$\le 2\pi|\beta-\alpha|\|g\|_{Lip(\alpha,\beta)}(1+|n||\beta-\alpha|)^{-1}\,.$$

We turn to the case $\frac{\pi}{n} \le \beta - \alpha$. We have

$$e^{in(x+\pi/n)} = -e^{inx}\,.$$

Using this, we write

$$\int_\alpha^\beta g(x)e^{inx}\,\mathrm{d}x = \frac{1}{2}\int_\alpha^\beta g(x)e^{inx}\,\mathrm{d}x - \frac{1}{2}\int_\alpha^\beta g(x)e^{in(x+\pi/n)})\,\mathrm{d}x\,.$$

We split the the first integral at $\alpha + \frac{\pi}{n}$ and the second one at $\beta - \frac{\pi}{n}$, and make a change of variables in the second part of the first integral to obtain

$$= \frac{1}{2}\int_\alpha^{\alpha+\frac{\pi}{n}} g(x)e^{inx}\,\mathrm{d}x - \frac{1}{2}\int_{\beta-\frac{\pi}{n}}^\beta g(x)e^{in(x+\pi/n)}\,\mathrm{d}x$$

$$+\frac{1}{2}\int_{\alpha+\frac{\pi}{n}}^\beta (g(x)-g(x-\frac{\pi}{n}))e^{inx}\,\mathrm{d}x\,.$$

The sum of the first two terms is by the triangle inequality bounded by

$$\frac{\pi}{n}\sup_{x\in[\alpha,\beta]}|g(x)|\,.$$

The third term is by the triangle inequality at most

$$\frac{1}{2}\int_{\alpha+\frac{\pi}{n}}^\beta |g(x)-g(x-\frac{\pi}{n})|\,\mathrm{d}x$$

$$\leq \frac{|\beta-\alpha|}{2}\frac{\pi}{n}\sup_{\alpha\leq x<y\leq\beta}\frac{|g(x)-g(y)|}{|x-y|}\,.$$

Adding the two terms, we obtain

$$\left|\int_\alpha^\beta g(x)e^{-inx}\,\mathrm{d}x\right| \leq \frac{\pi}{n}\|g\|_{\mathrm{Lip}(\alpha,\beta)}\,.$$

This completes the proof of the lemma, using that with $\frac{\pi}{n}\leq\beta-\alpha$,

$$\frac{\pi}{n} = \frac{2\pi|\beta-\alpha|}{2n|\beta-\alpha|} \leq 2\pi|\beta-\alpha|(1+n|\beta-\alpha|)^{-1}\,.$$

□

3.2.5 Partial sums as orthogonal projections

No changes have been made here compared to 10.7 of [Bec+24].

3.2.6 The error bound

Lemma 3.37 (Dirichlet kernel - Hilbert kernel relation)**.** *For all $N\in\mathbb{Z}$ and $x\in[-\pi,\pi]\setminus\{0\}$,*

$$\left|K_N(x)-(e^{-iNx}\kappa(x)+\overline{e^{-iNx}\kappa(x)})\right|\leq\pi\,.$$

Proof. Let $N\in\mathbb{Z}$ and $x\in[-\pi,\pi]\setminus\{0\}$. With Lemma 3.29, we obtain

$$K_N(x)-(e^{-iNx}\kappa(x)+\overline{e^{-iNx}\kappa(x)}) = e^{-iNx}\frac{\min(|x|,1)}{1-e^{ix}}+e^{iNx}\frac{\min(|x|,1)}{1-e^{-ix}}\,.$$

Using Lemma 3.30 with $\eta=\min(|x|,1)$, we bound

$$\left|K_N(x)-(e^{-iNx}\kappa(x)+\overline{e^{-iNx}\kappa(x)})\right|$$
$$\leq \frac{\min(|x|,1)}{|1-e^{ix}|}+\frac{\min(|x|,1)}{|1-e^{-ix}|}\leq\frac{\pi}{2}+\frac{\pi}{2}=\pi\,.$$

□

Lemma 3.38 (partial Fourier sum bound)**.** *Let $g:\mathbb{R}\to\mathbb{C}$ be a measurable 2π-periodic function such that for some $\delta>0$ and every $x\in\mathbb{R}$,*

$$|g(x)|\leq\delta\,. \tag{3.158}$$

Then for every $x\in[0,2\pi]$ and $N>0$,

$$|S_Ng(x)|\leq\frac{1}{2\pi}(Tg(x)+T\bar{g}(x))+\pi\delta.$$

Proof. Let $x \in [0, 2\pi]$ and $N > 0$. We have with Lemma 3.29

$$|S_N g(x)| = \frac{1}{2\pi} \left| \int_0^{2\pi} g(y) K_N(x-y)\, dy \right| .$$

We use 2π-periodicity of g and K_N to shift the domain of integration to obtain

$$= \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) K_N(x-y)\, dy \right| .$$

Using the triangle inequality, we split this as

$$\leq \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) \left(K_N(x-y) - \max(|x-y|, 0) K_N(x-y) \right) dy \right| \tag{3.159}$$

$$+ \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) \max(|x-y|, 0) K_N(x-y)\, dy \right| . \tag{3.160}$$

Note that all integrals are well defined, since K_N is by (3.142) bounded by $2N+1$. Using that

$$\max(|x-y|, 0) K_N(x-y) = e^{-iN(x-y)} \kappa(x-y) + \overline{e^{-iN(x-y)} \kappa(x-y)} , \tag{3.161}$$

Lemma 3.37 and (3.162), we bound (3.159) by

$$\frac{1}{2\pi} \int_{x-\pi}^{x+\pi} |g(y)| \left| K_N(x-y) - e^{-iN(x-y)} \kappa(x-y) + \overline{e^{-iN(x-y)} \kappa(x-y)} \right| dy$$

$$\leq \pi\delta .$$

By dominated convergence and since $\kappa(x-y) = 0$ for $|x-y| > 1$, (3.160) equals

$$\frac{1}{2\pi} \lim_{r \to 0^+} \left| \int_{r < |x-y| < 1} g(y) \max(|x-y|, 0) K_N(x-y)\, dy \right| .$$

We bound the limit by a supremum and rewrite using (3.161),

$$\leq \frac{1}{2\pi} \sup_{r>0} \left| \int_{r<|x-y|<1} g(y) \left(e^{-iN(x-y)} \kappa(x-y) + \overline{e^{-iN(x-y)} \kappa(x-y)} \right) dy \right|$$

Using the triangle inequality, we further bound this by

$$\leq \frac{1}{2\pi} \sup_{r>0} \left| \int_{r<|x-y|<1} g(y) e^{-iNy} \kappa(x-y)\, dy \right|$$

$$+ \frac{1}{2\pi} \sup_{r>0} \left| \int_{r<|x-y|<1} \overline{g}(y) e^{-iNy} \kappa(x-y)\, dy \right| .$$

By the definition (3.135) of T, this is

$$\leq \frac{1}{2\pi}(Tg(x) + T\bar{g}(x)) .$$

□

Lemma 3.39 (real Carleson operator measurable)**.** *Let f be a bounded measurable function on $\mathbb{R}$. Then Tf as defined in* (3.135) *is measurable.*

Proof. Since a countable supremum of measurable functions is measurable, it suffices to show that for every $n \in \mathbb{Z}$,

$$x \mapsto \sup_{r>0} \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right|$$

is measurable. So let $n \in \mathbb{Z}$. Note that for each $x \in \mathbb{R}$, the function

$$r \mapsto \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right|$$

is continuous on $(0, \infty)$ since the integrand is locally bounded on the domain $0 < |x - y| < 1$ by the assumptions on f and Lemma 3.32. Thus, for each $x \in \mathbb{R}$,

$$\sup_{r>0} \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right|$$
$$= \sup_{r\in\mathbb{Q}_{>0}} \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right|$$

The right hand side is again a countable supremum so it remains to show that for every $r \in \mathbb{Q}_{>0}$,

$$x \mapsto \left| \int \mathbf{1}_{\{r<|x-\cdot|<1\}}(y) f(y)\kappa(x-y)e^{iny}\, dy \right|$$

is measurable, which follows from the fact that the integrand is measurable in (x, y). □

Lemma 3.40 (partial Fourier sums of small)**.** *Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable 2π-periodic function such that for some $\delta > 0$ and every $x \in \mathbb{R}$,*

$$|g(x)| \leq \delta . \tag{3.162}$$

Then for every $\epsilon > 0$, there exists a measurable set $E \subset [0, 2\pi]$ with $|E| < \epsilon$ such that for every $x \in [0, 2\pi] \setminus E$ and $N > 0$,

$$|S_N g(x)| \leq C_\epsilon \delta, \tag{3.163}$$

where

$$C_\epsilon = \left(\frac{8}{\pi\epsilon}\right)^{\frac{1}{2}} C_{4,2} + \pi . \tag{3.164}$$

Proof. Define

$$E := \{x \in [0, 2\pi] \ : \ \sup_{N>0} |S_N g(x)| > C_\epsilon \delta\} \,.$$

Then (3.163) clearly holds, and it remains to show that $|E| \le \epsilon$. Using Lemma 3.38, we obtain

$$E \subset \{x \in [0, 2\pi] \ : \ C_\epsilon \delta < \frac{1}{2\pi}(Tg(x) + T\bar{g}(x)) + \pi\delta\} \subset E_1 \cup E_2,$$

where

$$\begin{aligned} E_1 :=& \{x \in [0, 2\pi] \ : \ \pi(C_\epsilon - \pi)\delta < Tg(x)\} \\ E_2 :=& \{x \in [0, 2\pi] \ : \ \pi(C_\epsilon - \pi)\delta < T\bar{g}(x)\}. \end{aligned}$$

By Lemma 3.39, E_1 and E_2 are measurable. Thus,

$$\pi(C_\epsilon - \pi)\delta |E_1| \le \int_{E_1} Tg(x)\, dx = \delta \int_{E_1} T(\delta^{-1} g \mathbf{1}_{[-\pi,3\pi]})(x)\, dx \,.$$

Applying Lemma 3.25 with $F = [-\pi, 3\pi]$ and $G = E'$, it follows that this is

$$\le \delta \cdot C_{4,2} |F|^{\frac{1}{2}} |E_1|^{\frac{1}{2}} \le (4\pi)^{\frac{1}{2}} C_{4,2} \delta \cdot |E'|^{\frac{1}{2}} \,.$$

Rearranging, we obtain

$$|E_1| \le \left(\frac{(4\pi)^{\frac{1}{2}} C_{4,2}}{\pi(C_\epsilon - \pi)} \right)^2 = \frac{\epsilon}{2} \,.$$

Analogously, we get the same estimate for $|E_2|$. This completes the proof using $|E| \le |E_1| + |E_2|$. □

Proof of Lemma 3.24. Lemma 3.24 now follows directly from Lemma 3.40 with $\delta := \epsilon'$. □

3.2.7 Carleson on the real line

No changes have been made here compared to 10.9 of [Bec+24].

4 Samples of the Formalization

We try to give a brief overview of design choices and statements of the formalization as well as difficulties encountered during the formalization.

We present a few selected Lean code samples. However, even for these we will not attempt to explain their syntax and semantics in full detail since this is beyond the scope of this thesis.[2] For the full Lean code in the context of the Carleson project, please see the code attached to this thesis which will come with a description which files the author has contributed.

Mathlib already contains some Fourier analysis basics which are mainly developed on the `AddCircle`. In our setting this will correspond to the quotient $\mathbb{R}/(2\pi\mathbb{Z})$ but the library allows arbitrary periods. Regarding consistency with mathlib, it might thus make sense to develop the theory of partial Fourier sums on the `AddCircle` as well. However, as the underlying blueprint sections were written exclusively from the perspective of the real line and since many of the arguments in chapter 10 of [Bec+24] concern non-periodic functions and thus might not immediately carry over to the `AddCircle` setting, it was decided to stick with this form for the purpose of this project. Another advantage of that choice is that mathlib has very good support for integrals over intervals on the real line. Nevertheless, now that a significant portion of the formalization has been done, it might be a worthwhile future project to translate it to the `AddCircle` setting.

That being said, we now present our definition of the partial Fourier sum in Lean. We could rely on mathlib for the definition of Fourier coefficients and the exponential monomials.

```
def partialFourierSum (N : ℕ) (f : ℝ → ℂ) (x : ℝ) : ℂ :=
  ∑ n ∈ Icc (-(N : ℤ)) N,
    fourierCoeffOn Real.two_pi_pos f n
      * fourier n (x : AddCircle (2 * Real.pi))
```

In `fourierCoeffOn Real.two_pi_pos f n`, the integration domain $(0, 2\pi)$ is encoded in `Real.two_pi_pos` which is a proof of `0 < 2 * Real.pi`, ensuring that this is indeed a non-empty interval. Since `fourier n` is a function on the `AddCircle`, we need to cast `x` accordingly.

Introducing the notation `S_` for `partialFourierSum`, we are ready to formulate Theorem 1.1 in Lean.

[2] We refer to the online resources
https://lean-lang.org/theorem_proving_in_lean4/,
https://lean-lang.org/lean4/doc/ and
https://leanprover-community.github.io/mathlib4_docs/.

```
theorem classical_carleson {f : ℝ → ℂ} (cont_f : Continuous f)
    (periodic_f : f.Periodic (2 * Real.pi))
    {ε : ℝ} (εpos : 0 < ε) :
  ∃ E ⊆ Set.Icc 0 (2 * Real.pi), MeasurableSet E ∧
  volume.real E ≤ ε ∧
  ∃ N₀, ∀ x ∈ (Set.Icc 0 (2 * Real.pi)) \ E, ∀ N > N₀,
    ‖f x - S_ N f x‖ ≤ ε
```

Believing that definitions and notation here mean what their name suggests, this indeed looks quite like the informal version. One thing to note here, because it also has some significance in other parts of the formalization, is the following. The name `volume` denotes the standard measure on $\mathbb{R}$. Measures assign values in $[0, \infty]$, the extended nonnegative real numbers. In Lean, this is the type `ℝ≥0∞`, distinct from the type `ℝ`, which means that elements of the different types cannot be directly compared to each other. There is also no automatic conversion (a so called *coercion*) here because there is no canonical embedding of one type into the other (although both can be embedded into the extended real numbers $[-\infty, \infty]$). The problem is solved here by using `volume.real` which assigns the value 0 where `volume` would take the value ∞. However, in general it should be checked carefully whether this changes the meaning of a statement. Here it is not the case as we require $E \subset [0, 2\pi]$ anyway.

The definition of the generalized Carleson operator and the statements of the general Theorems 1.2 and 3.2 have not been formalized by the author but during the course of formalizing the deduction of Theorem 1.1, minor issues with the original versions similar to the ones described above have been discovered and corrected. The supremum (denoted `⨆`) of terms of type `ℝ` has type `ℝ`, again taking the value 0 when it actually should be ∞. The solution in this case was to first cast the elements to `ℝ≥0∞` and then take the supremum within this type.

```
def CarlesonOperator [FunctionDistances ℝ X]
  (K : X → X → ℂ) (f : X → ℂ) (x : X) : ℝ≥0∞ :=
   ⨆ (Q : Θ X) (R₁ : ℝ) (R₂ : ℝ) (_ : 0 < R₁) (_ : R₁ < R₂),
     ↑‖∫ y in {y | dist x y ∈ Ioo R₁ R₂},
        K x y * f y * exp (I * Q y)‖₊
```

The default integral in mathlib is the Bochner integral. However, it is only defined for functions taking values in a Banach space and will assume the value 0 when its integrand is not integrable.

To express the kind of estimate for integrals of nonnegative functions (possibly taking the value ∞) just stating a bound and implying integrability as in Theorems 1.2 and 3.2, we need the notion of the lower Lebesgue integral, denoted `∫⁻` in Lean.

```
theorem two_sided_metric_carleson
    [CompatibleFunctions ℝ X (2 ^ a)]
    [IsCancellative X (defaultτ a)] [IsTwoSidedKernel a K]
    (ha : 4 ≤ a) (hq : q ∈ Ioc 1 2) (hqq' : q.IsConjExponent q')
    (hF : MeasurableSet F) (hG : MeasurableSet G)
    (hT : ∀ r > 0,
      HasBoundedStrongType (CZOperator K r)
        2 2 volume volume (C_Ts a))
    (f : X → ℂ) (hf : ∀ x, ‖f x‖ ≤ F.indicator 1 x) :
  ∫⁻ x in G, CarlesonOperator K f x ≤
    ENNReal.ofReal (C10_1 a q)
      * (volume G) ^ q'⁻¹ * (volume F) ^ q⁻¹
```

In the personal experience of the author, the issues named above were among the most notable translation difficulties from the informal world to Lean in the specific area that has been worked on. The common pattern of assigning default values to expressions that in the informal world would be considered ill-formed and having *no value at all* can also be observed in the example $0^{-1} = 0$. These definitions can help to avoid considering corner cases sometimes and they delay the necessity to prove well-definedness (in the sense of not being in the case of the default value) to the point where a manipulation of an expression actually requires it to be valid. However, in the context of many consecutive manipulations of integrals, this often has the disadvantage that integrability of multiple very similar functions needs to be shown.

On the other hand, at least in the author's experience most additional arguments in the formal proofs originate from omissions in the informal proof. An outstanding example here is measurability of functions, which is obviated in many places of the blueprint. The hidden proofs vary from truely obvious, in which case they can often be solved by the tactic `measurability` or at least in one or two lines of code, to requiring more than fifty lines of code.

We conclude with the remark that, with an already very detailed blueprint available, none of the minor difficulties presented here poses an insurmountable obstacle to formalization and that formalizing larger parts of foundational as well as cutting-edge mathematics is becoming increasingly realistic with the fast growth of mathlib and the development of ever more powerful automation.

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