Primes, Proofs and Computers

Antrittsvorlesung

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Computers in science and math

Computers are used extensively in science:

- To compute
- To simulate
- To record data
- To perform statistical analysis
- To write papers and books
- To exchange ideas online

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Mathematicians prove theorems about abstract mathematical concepts.

However, computers are rarely used for finding or checking proofs.

Formalization

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The first proof assistant Automath was developed by Dutch Mathematician De Bruijn in 1968.

A proof assistant that is popular among mathematicians is Lean, an open source program in development since 2013.

Overview of the talk

- Infinitude of primes: Euclid's proof
- Infinitude of primes: proof in Lean
- More on formalization

Primes

Definition

A natural number $(0, 1, 2, \ldots)$ is prime if it is greater than 1 and it cannot be written as the product of two smaller natural numbers.

The first few prime numbers are $2, 3, 5, 7, 11, 13, \ldots$
117 is not prime, since $117 = 3 \times 39 = 3 \times 3 \times 13$.
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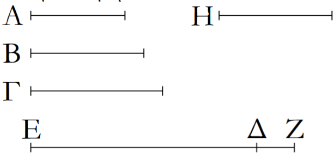
Theorem (Euclid)

There are infinitely many prime numbers.

κ΄.

Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν.



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Εἰλήφθω γὰρ ὁ ὑπὸ τῶν Α, Β, Γ ἐλάχιστος μετρούμενος καὶ ἔστω ΔΕ, καὶ προσκείσθω τῷ ΔΕ μονὰς ἡ ΔΖ. ὁ δὴ ΕΖ ἤτοι πρῶτός ἐστιν ἢ οὔ. ἔστω πρότερον πρῶτος· εὑρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ οἱ Α, Β, Γ, ΕΖ πλείους τῶν Α, Β, Γ.

Ἀλλὰ δὴ μὴ ἔστω ὁ ΕΖ πρῶτος· ὑπὸ πρώτου ἄρα τινὸς ἀριθμοῦ μετρεῖται. μετρείσθω ὑπὸ πρώτου τοῦ Η· λέγω, ὅτι ὁ Η οὐδενὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. εἰ γὰρ δυνατόν, ἔστω. οἱ δὲ Α, Β, Γ τὸν ΔΕ μετροῦσιν· καὶ ὁ Η ἄρα τὸν ΔΕ μετρήσει. μετρεῖ δὲ καὶ τὸν ΕΖ· καὶ λοιπὴν τὴν ΔΖ μονάδα μετρήσει ὁ Η ἀριθμὸς ὤν· ὅπερ ἄτοπον. οὐκ ἄρα ὁ Η ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. καὶ ὑπόκειται πρῶτος. εὑρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ πλείους τοῦ προτεθέντος πλήθους τῶν Α, Β, Γ οἱ Α, Β, Γ, Η· ὅπερ ἔδει δεῖξαι.

(Book IX, Proposition 20)

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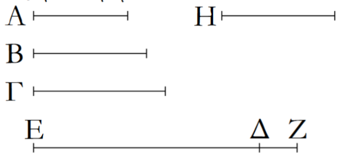

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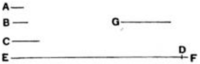
PROPOSITION 20.

Prime numbers are more than any assigned multitude of prime numbers.

Let *A*, *B*, *C* be the assigned prime numbers;

I say that there are more prime numbers than *A*, *B*, *C*.



For let the least number measured by *A*, *B*, *C* be taken,

and let it be *DE*;

let the unit *DF* be added to *DE*.

Then *EF* is either prime or not.

First, let it be prime;

then the prime numbers *A*, *B*, *C*, *EF* have been found which are more than *A*, *B*, *C*.

Next, let *EF* not be prime;

therefore it is measured by some prime number. [VII. 31]

Let it be measured by the prime number *G*.

I say that *G* is not the same with any of the numbers *A*, *B*, *C*.

For, if possible, let it be so.

Now *A*, *B*, *C* measure *DE*;

therefore *G* also will measure *DE*.

But it also measures *EF*.

Therefore *G*, being a number, will measure the remainder, the unit *DF*:

which is absurd.

Therefore *G* is not the same with any one of the numbers *A*, *B*, *C*.

And by hypothesis it is prime.

Therefore the prime numbers *A*, *B*, *C*, *G* have been found which are more than the assigned multitude of *A*, *B*, *C*.

Q. E. D.

(translation by Thomas Little Heath, 1908)

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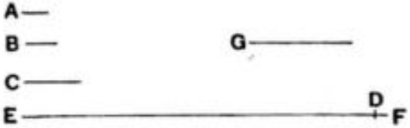

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Proof in Lean

Demo:

```
theorem Euclid
  (P : finite subset of ℕ)
  (hP : for all p in P, p is prime) :
  exists p,
  p is prime and p is not in P
```

Lean's mathematical library

Lean has a mathematical library with results from many fields in mathematics:
algebra, analysis, geometry, probability theory, combinatorics, logic, topology, category theory, ...

It is large: Mathlib has over 1 million lines of code, with thousands of definitions and theorems written by over 300 contributors.

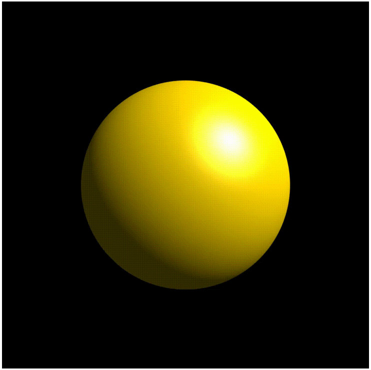
It is actively developed: There are more than 100 contributions every week, reviewed by the maintainers.

Sphere eversion

Can we turn a sphere inside out?

Rules:

- No tears or sharp creases;
- It is allowed to self-intersect.



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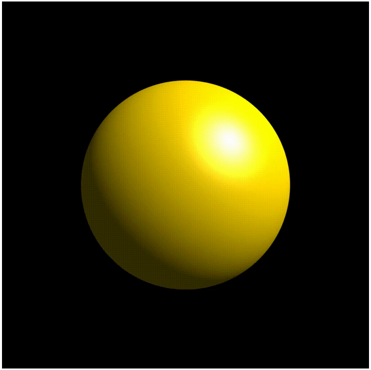

Sphere eversion

Theorem (Smale, 1957)

There is a smooth transformation of immersions

$$\mathbb{S}^2 \to \mathbb{R}^3$$

from the inclusion map to the antipodal map.

This follows from a deep result in differential geometry.

Theorem (Gromov, 1973)

If $\mathcal{R}$ is an open and ample partial differential relation for functions between manifolds M and N then $\mathcal{R}$ satisfies the homotopy principle.

In 2022 I formalized these theorems in Lean together with Patrick Massot and Oliver Nash.

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1. Collaboratively create a unified mathematical library
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4. Create new mathematics
   - Being forced to write rigorous proofs leads one to gain new insights in mathematics.
   - Potentially in the future by AI.

Thanks for listening

Questions?

Sphere eversion, formalized

```
theorem sphere_eversion :
  ∃ f : ℝ → 𝕊² → ℝ³,
  smooth (↿f : ℝ × 𝕊² → ℝ³) ∧
  f 0 = (coe : 𝕊² → ℝ³) ∧
  f 1 = (-coe : 𝕊² → ℝ³) ∧
  ∀ t, immersion (f t)
```

The homotopy principle

Theorem (Gromov, 1973)

If $\mathcal{R}$ is an open and ample[1] *partial differential relation for functions between manifolds M and N then $\mathcal{R}$ satisfies the homotopy principle, i.e. any formal solution can be smoothly deformed into a holonomic one inside $\mathcal{R}$.*

[1] Ampleness is a geometric condition that ensures that certain convex hulls are large enough for the convex integration argument to work.