Spectral Sequences in Homotopy Type Theory

Floris van Doorn

Carnegie Mellon University

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Joint work with Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke and Mike Shulman.

Outline

- Cohomology in HoTT
- Spectral sequences
- Atiyah-Hirzebruch and Serre spectral sequences for cohomology
- Future work: Spectral sequences for homology

Cohomology

How do we define cohomology?

The usual constructions of singular cohomology is not homotopy invariant.

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Classical theorem: The Eilenberg-MacLane spaces $K(A, n)$ classify cohomology.

> Recall. The Eilenberg-MacLane space $K(A, n)$ is the unique pointed type X with one nontrivial homotopy group $\pi_n(X) \simeq A$.

Cohomology

We define the reduced cohomology of a pointed type X with coefficients in an abelian group A to be

$$\widetilde{H}^n(X, A) :\equiv \|X \to^* K(A, n)\|_0.$$

The unreduced cohomology can be defined similarly for any (not necessarily pointed) type X:

$$H^n(X, A) :\equiv \|X \to K(A, n)\|_0 = \widetilde{H}^n(X + 1, A).$$

The group structure comes from the equivalence $K(A, n) \simeq^* \Omega K(A, n + 1)$.

Long Exact Sequence of Homotopy Groups

Given a pointed map $f : X \to^* Y$ with fiber F.
Then we have the following long exact sequence.

$$
\begin{array}{ccccc}
 & & \vdots & & \\
\pi_2(F) & \xrightarrow[\pi_2(p_1)]{} & \pi_2(X) & \xrightarrow{\pi_2(f)} & \pi_2(Y) \\
 & \swarrow_{\pi_1(\delta)} & & & \\
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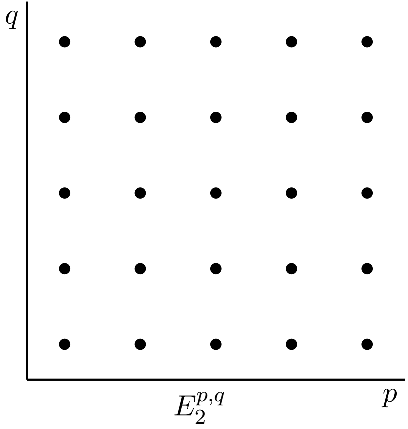
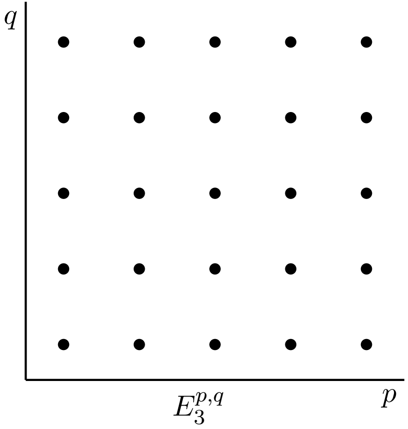
\pi_0(F) & \xrightarrow[\pi_0(p_1)]{} & \pi_0(X) & \xrightarrow{\pi_0(f)} & \pi_0(Y)
\end{array}
$$

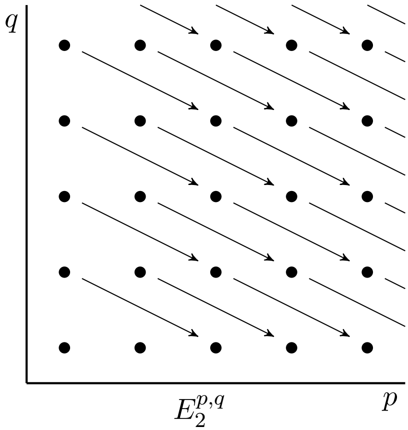
Spectral Sequences

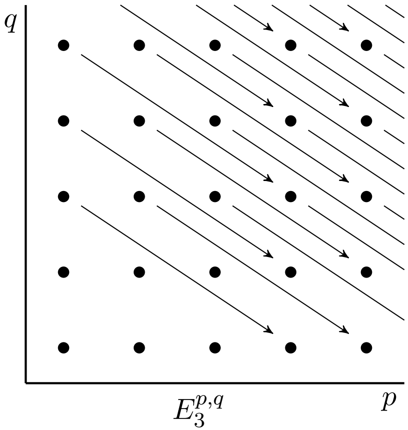
Definition A spectral sequence consists of a family $E_r^{p,q}$ of abelian groups for $p, q : \mathbb{Z}$ and $r \geq 2$. For a fixed r this gives the r-*page* of the spectral sequence. ...



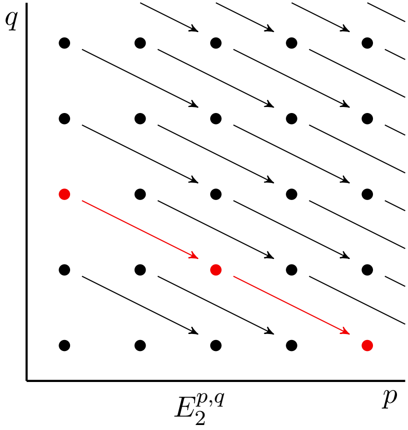


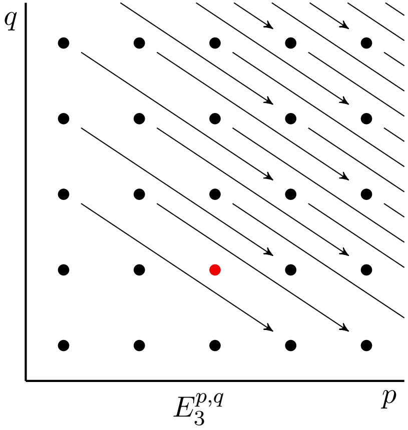
Definition ... with *differentials* $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$ (this is cohomologically indexed) ...





Definition … and with isomorphisms $\alpha_r^{p,q} : H^{p,q}(E_r) \simeq E_{r+1}^{p,q}$ where $H^{p,q}(E_r) = \ker(d_r^{p,q})/\operatorname{im}(d_r^{p-r,q+r-1})$.





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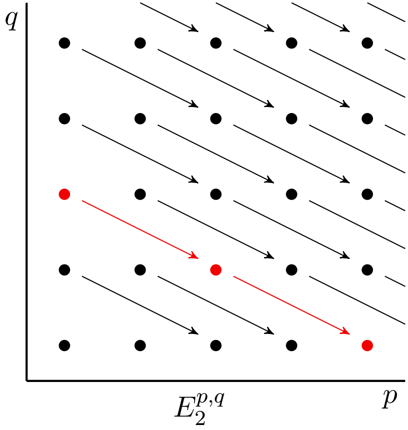
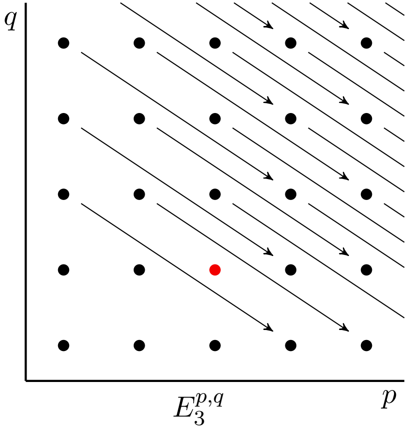
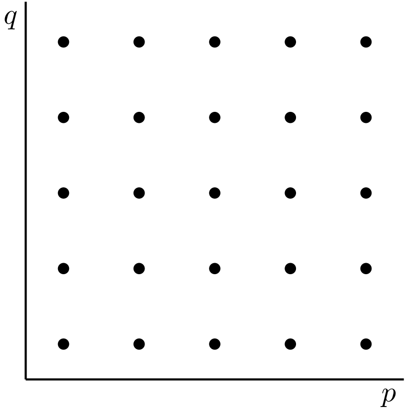
The differentials of E_{r+1} are not determined by E_r.





Convergence of Spectral Sequences

In many spectral sequences the pages *converge* to $E_\infty^{p,q}$.



$E_\infty^{p,q}$

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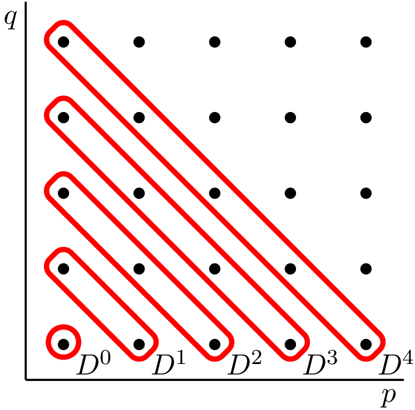
And we can get information about the diagonals on the infinity page.



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  We have short exact sequences:

$$\begin{gathered}
0 \to E_\infty^{0,n} \to D^n \to D^{n,1} \to 0 \\
\vdots \\
0 \to E_\infty^{p,q} \to D^{n,p} \to D^{n,p+1} \to 0 \\
0 \to E_\infty^{p+1,q-1} \to D^{n,p+1} \to D^{n,p+2} \to 0 \\
\vdots \\
0 \to E_\infty^{n,0} \to D^{n,n} \to 0
\end{gathered}$$

Serre Spectral Sequence (Special Case)

Theorem. Suppose $f : X \to B$ and $b_0 : B$ and let $F :\equiv \mathsf{fib}_f(b_0)$. Suppose that B is simply connected and A is an abelian group. Then

$$E_2^{p,q} = H^p(B, H^q(F, A)) \Rightarrow H^{p+q}(X, A).$$

This is only true for unreduced cohomology.

Example: Cohomology of $K(\mathbb{Z}, 2)$

We will compute the cohomology groups of $B = K(\mathbb{Z}, 2)$ (which is $\mathbf{CP}^\infty$).
We define the map $1 \xrightarrow{f} K(\mathbb{Z}, 2)$ determined by the basepoint $\star : K(\mathbb{Z}, 2)$.
It has fiber

$$\mathsf{fib}_f(\star) = \Omega K(\mathbb{Z}, 2) = K(\mathbb{Z}, 1) = \mathbb{S}^1.$$

The spectral sequence for $A = \mathbb{Z}$ gives

$$E_2^{p,q} = H^p(B, H^q(\mathbb{S}^1)) \Rightarrow H^{p+q}(1).$$

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$$H^n(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \qquad H^n(1) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

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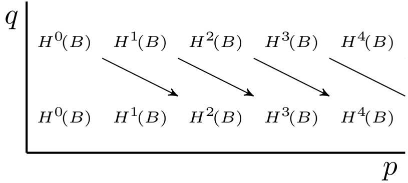

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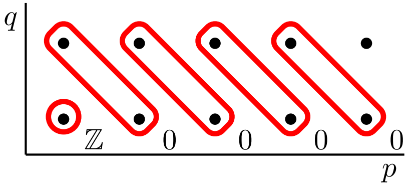

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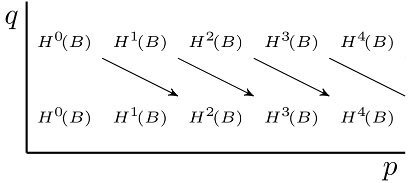

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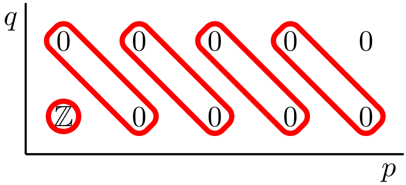

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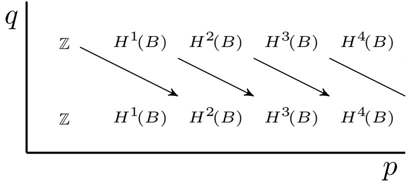

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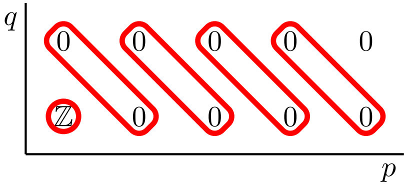

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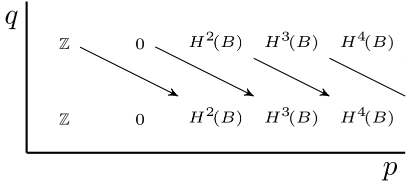

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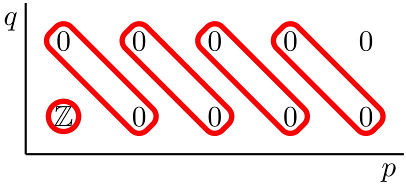

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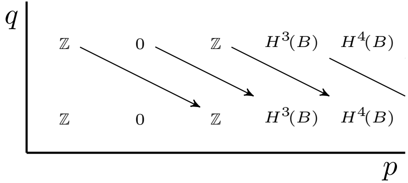

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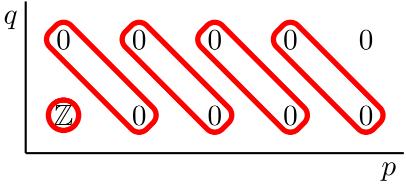

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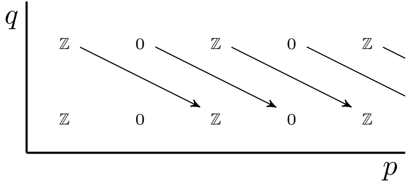

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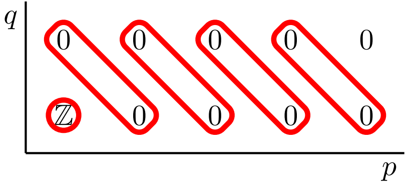

$E_\infty^{p,q}$

Spectra

For the general Serre spectral sequence, we need generalized and parametrized cohomology.

An (Ω-)spectrum is a sequence of pointed types $Y : \mathbb{Z} \to \mathrm{Type}^*$ such that $\Omega Y_{n+1} = Y_n$.

Example. If A is an abelian group, $HA : \mathrm{Spectrum}$ where $(HA)_n = K(A, n)$.

A spectrum Y is called n-truncated if Y_k is $(n+k)$-truncated for all $k : \mathbb{Z}$.

The homotopy groups are $\pi_n(Y) :\equiv \pi_{n+k}(Y_k)$ (which is independent of k and also defined for negative n).

Generalized Cohomology

If X is a type and Y is a spectrum, we have generalized cohomology:

$$H^n(X, Y) :\equiv \|X \to Y_n\|_0 \simeq \pi_{-n}(X \to Y).$$

We get generalized and parametrized cohomology by replacing functions with dependent functions:

$$H^n(X, \lambda x.Yx) :\equiv \|\Pi(x : X),\ Y_n(x)\|_0. \simeq \pi_{-n}(\Pi(x : X),\ Yx)$$

Here X is a type and $Y : X \to \mathrm{Spectrum}$.

Reduced cohomology is defined similar with basepoint-preserving sections.

Serre Spectral Sequence

Theorem. *(Serre Spectral Sequence)* If $f : X \to B$ is any map and Y is a truncated spectrum, then

$$E_2^{p,q} = H^p(B, \lambda b.H^q(\mathsf{fib}_f(b), Y)) \Rightarrow H^{p+q}(X, Y).$$

If $Y = HA$ and B is pointed simply connected, then this reduces to the previous case

$$E_2^{p,q} = H^p(B, H^q(\mathsf{fib}_f(b_0), A)) \Rightarrow H^{p+q}(X, A).$$

Atiyah-Hirzebruch Spectral Sequence

Theorem. *(Atiyah-Hirzebruch Spectral Sequence)* If X is any type and $Y : X \to \mathrm{Spectrum}$ is a family of k-truncated spectra over X, then

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The Atiyah-Hirzebruch spectral sequence is also true if we replace all cohomologies by reduced cohomologies:

$$E_2^{p,q} = \widetilde{H}^p(X, \lambda x.\pi_{-q}(Yx)) \Rightarrow \widetilde{H}^{p+q}(X, \lambda x.Yx).$$

Construction (1)

Based on the construction (sketch) by Shulman

[`ncatlab.org/homotopytypetheory/show/spectral+sequences`]

For pointed types we have the fiber sequence

$$K(\pi_k(Z), k) \longrightarrow \|Z\|_k \longrightarrow \|Z\|_{k-1}.$$

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Given $X : \mathrm{Type}^*$ and $Y : X \to \mathrm{Spectrum}$ which are s_0-truncated. For $x : X$ and $s : \mathbb{Z}$ we have the following fiber sequence of spectra:

$$K(\pi_s(Yx), s) \longrightarrow \|Yx\|_s \longrightarrow \|Yx\|_{s-1}.$$

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The functor $Y \mapsto \Pi^*(x : X),\ Yx$ preserves fiber sequences:

$$\Pi^*(x : X),\ K(\pi_s(Yx), s) \longrightarrow \Pi(x : X),\ \|Yx\|_s \longrightarrow \Pi^*(x : X),\ \|Yx\|_{s-1}.$$

Let's call these types B_s and A_s:

$$B_s \longrightarrow A_s \longrightarrow A_{s-1}.$$

Construction (2)

$$B_s \longrightarrow A_s \longrightarrow A_{s-1}.$$

The long exact sequence of this fiber sequence gives:

$$\begin{array}{ccccc}
 & & \vdots & & \\
\pi_{n+1}(B_s) & \xrightarrow[k]{} & \pi_{n+1}(A_s) & \xrightarrow{i} & \pi_{n+1}(A_{s-1}) \\
 & & \swarrow_{j} & & \\
\pi_{n}(B_s) & \xrightarrow[k]{} & \pi_{n}(A_s) & \xrightarrow{i} & \pi_{n}(A_{s-1}) \\
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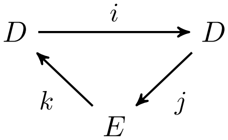
\pi_{n-1}(B_s) & \xrightarrow[k]{} & \pi_{n-1}(A_s) & \xrightarrow{i} & \pi_{n-1}(A_{s-1}) \\
 & & \vdots & &
\end{array}$$

Construction (3)

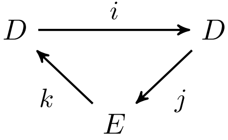
Define

$$E = \bigoplus_{n,s} \pi_n(B_s) \quad \text{and} \quad D = \bigoplus_{n,s} \pi_n(A_s).$$

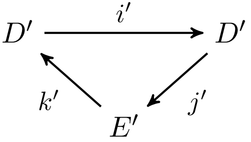
These long exact sequences give an *exact couple* between bigraded abelian groups.



From an exact couple



we build a *derived exact couple*



where E' is the (co)homology of E with differential $d :\equiv j \circ k : E \to E$.

Construction (5)

We iterate this process, so that we get a sequence of exact couples $(E_r, D_r, i_r, j_r, k_r)$.

Now $(E_r, d_r)_r$ forms the Atiyah-Hirzebruch spectral sequence:

$$E_2^{p,q} = \widetilde{H}^p(X, \lambda x.\pi_{-q}(Yx)) \Rightarrow \widetilde{H}^{p+q}(X, \lambda x.Yx).$$

(We have applied the reindexing $(p, q) = (s - n, -s)$.)

Construction (6)

$$E_2^{p,q} = H^p(X, \lambda x.\pi_{-q}(Yx)) \Rightarrow H^{p+q}(X, \lambda x.Yx).$$

$$E_2^{p,q} = H^p(B, \lambda b.H^q(\mathsf{fib}_f(b), Z)) \Rightarrow H^{p+q}(X, Z).$$

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For the Serre spectral sequence, we're given a map $f : X \to B$ and a truncated spectrum Z. We define

$$Y = \lambda(b : B).\mathsf{fib}_f(b) \to Z : B \to \mathrm{Spectrum}\,.$$

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Then

$$\pi_{-q}(Yb) = \pi_{-q}(\mathsf{fib}_f(b) \to Z) = H^q(\mathsf{fib}_f(b), Z)$$

$$\begin{aligned} H^{p+q}(B, \lambda b.Yb) &= \pi_{-(p+q)}(\Pi(b : B),\ \mathsf{fib}_f(b) \to Z) \\ &= \pi_{-(p+q)}((\Sigma(b : B),\ \mathsf{fib}_f(b)) \to Z) \\ &= \pi_{-(p+q)}(X \to Z) \\ &= H^{p+q}(X, Z) \end{aligned}$$

This gives the Serre spectral sequence

$$E_2^{p,q} = H^p(B, \lambda b.H^q(\mathsf{fib}_f(b), Z)) \Rightarrow H^{p+q}(X, Z).$$

Formalization

- Construction formalized in the Lean proof assistant.
- Available at `github.com/cmu-phil/Spectral`.
- The formalization took almost 2 years: November 2015 – July 2017.
- Formalized by vD, Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke and Mike Shulman.
- Formalization is $\sim$10k-20k LoC (the total size of Lean-HoTT is 53k LoC).

Applications

The remainder of the slides is mostly future work.

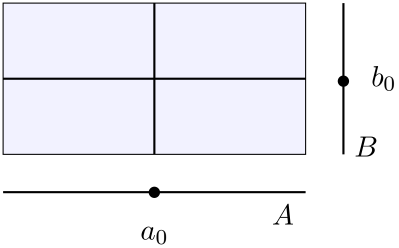
- We can compute cohomology groups of certain spaces (such as $K(\mathbb{Z}, n)$ and ΩS^n).
- We can compute cohomology groups of generalized cohomology theories (K-theory).
- We can construct the Gysin and Wang sequences.

To compute more homotopy groups of spheres, we need the Serre spectral sequence for homology.

Smash Product

For pointed types A and B, the smash product $A \wedge B$ is the following homotopy pushout.

$$\begin{array}{ccc} A \vee B & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ A \times B & \longrightarrow & A \wedge B \end{array}$$



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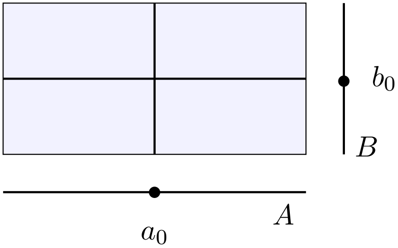


Homology with coefficients in a spectrum Y can be defined as

$$\widetilde{H}_n(X, Y) = \pi_n(X \wedge Y) = \mathsf{colim}_k(\pi_{n+k}(X \wedge Y_k)).$$

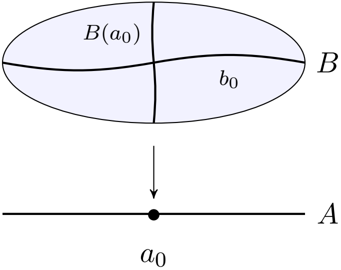
Parametrized Homology

We will also need parametrized homology.

$$\widetilde{H}_n(X, \lambda x.Yx) :\equiv \pi_n((x : X) \wedge Yx)$$

$(x : A) \wedge B(x)$ is a parametrized version of the smash product, the following homotopy pushout:

$$\begin{array}{ccc} A \vee B(a_0) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \Sigma(x : A),\ B(x) & \longrightarrow & (x : A) \wedge B(x) \end{array}$$



Spectral Sequences for Homology

Some challenges:

- Smashing doesn't preserve spectra: we need to apply spectrification.
- We need to prove that smashing preserves fiber sequences.

We should get the corresponding spectral sequences for homology:

$$E^2_{p,q} = \widetilde{H}_p(X, \lambda x.\pi_q(Yx)) \Rightarrow \widetilde{H}_{p+q}(X, \lambda x.Yx).$$

$$E^2_{p,q} = H_p(B, \lambda b.H_q(\mathsf{fib}_f(b), Y)) \Rightarrow H_{p+q}(X, Y).$$

Applications

Applications of the homology Serre spectral sequence:

- Serre class theorem (constructively?)
- Hurewicz theorem
- Computation of $\pi_{n+k}(\mathbb{S}^n)$ for $k \leq 3$.