Applications of the Serre Spectral Sequence

Floris van Doorn

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1 Serre Spectral Sequence

Definition 1.1. A *Spectral Sequence* is a sequence $(E^r_{p,q}, d_r)$ consisting of

- An R-module $E^r_{p,q}$ for $p, q \in \mathbb{Z}$ and $r \geq 0$.
- Differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $d_r^2 = 0$

where E^{r+1} is defined to be the homology of (E^r, d^r). That is, $E^{r+1}_{p,q} = \ker(d^r_{p,q}) / \operatorname{im}(d^r_{p+r,q-r+1})$. The variable r is called the *page*, p the *filtration degree*, q the *complementary degree* and $p+q$ the *total degree*.

Theorem 1.2 (Serre Spectral Sequence)**.** *Let $F \to X \twoheadrightarrow B$ be a fibration such that B is path-connected and $\pi_1(B)$ acts trivially on $H_*(F; G)$. Then*

$$H_p(B; H_q(F; G)) \implies H_{p+q}(X; G).$$

This means that there is a spectral sequence $(E^r_{p,q}, d_r)$ where $E^2_{p,q} \simeq H_p(B; H_q(F; G))$ and there is a filtration $0 \subseteq F^0_{p+q} \subseteq \cdots \subseteq F^{p+q}_{p+q} = H_{p+q}(X; G)$ such that $E^\infty_{p,q} \simeq F^p_{p+q}/F^{p-1}_{p+q}$.

Note that if B is simply connected, then conditions of the theorem are satisfied.

1.1 Examples

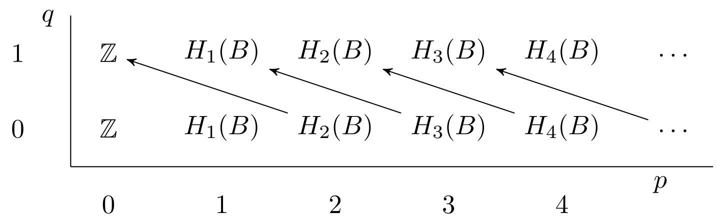
Example 1.3. Suppose $X = B \times F$, where B is path-connected, and suppose that G is a field. Then $\pi_1(B)$ acts trivially on $H_*(F; G)$ and we have

$$\begin{aligned} H_n(X; G) &= \bigoplus_{p+q=n} H_p(B; G) \otimes H_q(F; G) && \text{(Künneth formula)} \\ &= \bigoplus_{p+q=n} H_p(B; H_q(F; G)) && \text{(Univ. Coeff. Th. for homology)} \end{aligned}$$

This means that all entries in the second page survive until page infinity. The other extreme is if X is contractible, where almost nothing will survive, as we will see in the next examples. △

In the next example, we will use that $S^1 = K(\mathbb{Z}, 1)$ and $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

Example 1.4. Consider the path space fibration of $B = \mathbb{CP}^\infty$, that is $\Omega B \to PB \twoheadrightarrow B$ and note that $S^1 = \Omega B$. Since B is simply connected, we can apply the Serre Spectral Sequence with coefficients in $\mathbb{Z}$. We know that $E^2_{p,q} \simeq H_p(B; H_q(S^1))$ and $H_q(S^1) = 0$ for $q > 1$ and $\mathbb{Z}$ for $q = 0, 1$. This means that the page E^2 looks like this.

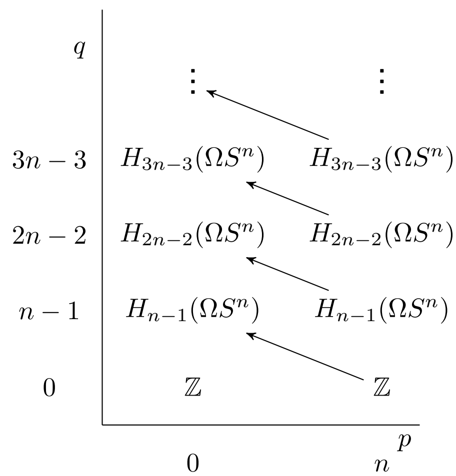


Moreover, we have $E^{\infty}_{p,q} = \mathbb{Z}$ for $p = q = 0$ and 0 otherwise. From this we can conclude that

$$H_i(\mathbb{C}\mathrm{P}^{\infty}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd.} \end{cases}$$

△

Example 1.5. In this example we will compute the homology groups of the loop space of the sphere, ΩS^n for $n \geq 2$. We use the fibration $\Omega S^n \to PS^n \twoheadrightarrow S^n$ and we can apply the Serre Spectral Sequence, since S^n is simply connected. Now $H_p(S^n; G) = G$ for $p = 0, n$ and 0 otherwise. This means that only the 0 and the n column can be nonzero.



After some reasoning, we get that $H_i(\Omega S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n-1 \mid i \\ 0 & \text{otherwise.} \end{cases}$ △

2 Serre Class Theorem

Definition 2.1. We say that a space X is *abelian* if the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geq 1$.

Note that every simply connected space is abelian.

Definition 2.2. A *Serre Class* is a class $\mathcal{C}$ of abelian groups containing the trivial group such that for every SES $0 \to A \to B \to C \to 0$ we have $B \in \mathcal{C}$ iff $A, C \in \mathcal{C}$. In this document I call a Serre class *nice* if for every $A, B \in \mathcal{C}$ also $A \otimes B$ and $\mathrm{Tor}(A, B)$ are in $\mathcal{C}$. (this name is made up by me)

Lemma 2.3. *The following classes are nice Serre classes.*

- *$\mathcal{FG}$, the class of finitely generated abelian groups*

- *$\mathcal{T}_P$ for some set P of primes. This is the class of torsion abelian groups whose elements have orders divisible only by primes in P.*
- *$\mathcal{F}_P$, the finite groups in $\mathcal{T}_P$.*

Note that P is the set of all primes $\mathcal{T}_P$ becomes the class of all torsion abelian groups and $\mathcal{F}_P$ becomes the class of all finite abelian groups.

Theorem 2.4. *Let X be a path-connected and abelian space, and let $\mathcal{C}$ be a nice Serre class. Then*

$$\forall(n > 0)(\pi_n(X) \in \mathcal{C}) \quad \longleftrightarrow \quad \forall(n > 0)(H_n(X) \in \mathcal{C})$$

Corollary 2.5. *The homotopy groups of a finite simply connected CW-complex are finitely generated. In particular, the homotopy groups of spheres are finitely generated.*

Recall the following definition and theorem.

Definition 2.6. The *Hurewicz homomorphism* is the homomorphism $h : \pi_n(X) \to H_n(X)$ defined by $h([f]) = f_*(\gamma)$, where γ is a generator of $H_n(S^n) \simeq \mathbb{Z}$.

Theorem 2.7 (Hurewicz)**.** *Let $n \geq 2$ and X a $(n-1)$-connected space. Then $\widetilde{H}_i(X) = 0$ for $i < n$ and the Hurewicz homomorphism $h : \pi_n(X) \to H_n(X)$ is an isomorphism.*

We will now generalize this theorem.

Theorem 2.8. *Let X be a path-connected and abelian space, and let $\mathcal{C}$ be a nice Serre class. Suppose that $\pi_i(X) \in \mathcal{C}$ for $i < n$. Then the Hurewicz homomorphism $h : \pi_n(X) \to H_n(X)$ is an isomorphism modulo $\mathcal{C}$, which means that its kernel and cokernel are in $\mathcal{C}$.*

3 Cohomology Serre Spectral Sequence

There is a Serre Spectral Sequence for cohomology which is completely analogous. Of course, the arrows in these spectral sequences are reversed.

Theorem 3.1. *Let $F \to X \twoheadrightarrow B$ be a fibration such that B is path-connected and $\pi_1(B)$ acts trivially on $H_*(F;G)$. Then*

$$H^p(B; H^q(F;G)) \implies H^{p+q}(X;G).$$

However, we can now also use the cup product if the underlying group is a ring.

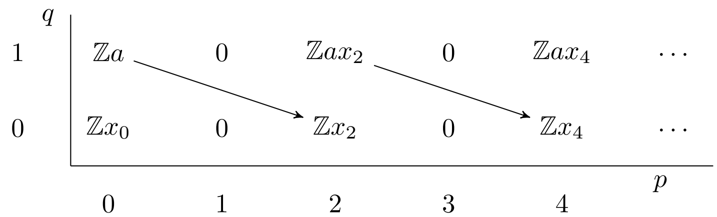
Theorem 3.2. *There is a bilinear product $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ for $1 \leq r \leq \infty$ (written as concatenation) satisfying*

- *For $x \in E_r^{p,q}$ we define $|x| = p + q$. Then we have $d_r(xy) = (d_r x)y + (-1)^{|x|}x(d_r y)$. This means that the product on level r induces a product on level $r+1$, which coincides with the given bilinear product at level $r+1$. The product in E_∞ is induces from the products at the finite levels.*
- *At page 2, the product is up to a factor $(-1)^{qs}$ induced from the cup product under the correspondence $E_2^{p,q} \simeq H^p(B; H^q(F;R))$. In the RHS the cup product sends (ϕ, ψ) to $\phi \smile \psi$ and coefficients are also multiplied via the cup product.*
- *The cup product in $H^*(X;R)$ restricts to maps on the filtrations $F_p^m \times F_s^n \to F_{p+s}^{m+n}$ which induce quotient maps $F_p^m / F_{p+1}^m \times F_s^n / F_{s+1}^n \to F_{p+s}^{m+n} / F_{p+s+1}^{m+n}$. Under the correspondence $E_\infty^{p,q} \simeq F_p^{p+q} / F_{p+1}^{p+q}$ the product in the LHS corresponds to these maps in the RHS.*
- *$ab = (-1)^{|a||b|}ba$ and $d(x^n) = nx^{n-1}dx$ if $|x|$ is even.*

In particular, if we apply the cup $E_2^{p,0} \times E_2^{0,q} \to E_2^{p,q}$ to a pair of units, we get a unit.

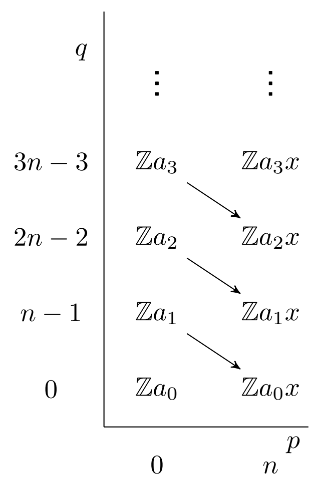
3.1 Examples

Example 3.3. Consider the path space fibration of $B = \mathbb{C}\mathrm{P}^\infty$ again. By the universal coefficient theorem, we know that the cohomology groups are the same as the homology groups. Now let's compute the cup product structure. Let x_{2i} be the generator of $E_2^{2i,0}$ and a the generator of $E_2^{0,1}$. Then x_0 is the unit for multiplication, and ax_{2i} are generators of $E_2^{2i,1}$.



All arrows are isomorphisms. We may assume that $d_2 a = x_2$. Then we compute $d_2(ax_{2i}) = x_2 x_{2i}$ so we may assume that $x_2 x_{2i} = x_{2i+2}$. This gives $x_{2i} = x_2^i$. Hence $H^*(\mathbb{C}\mathrm{P}^\infty, \mathbb{Z}) \simeq \mathbb{Z}[x_2]$. △

Example 3.4. We will compute the cup product structure of $H^*(\Omega S^n; \mathbb{Z})$ using the path space fibration of S^n for $n \geq 2$. The additive structure is the same as for homology, and we can name the generators as in the figure, where $a_0 = 1$.

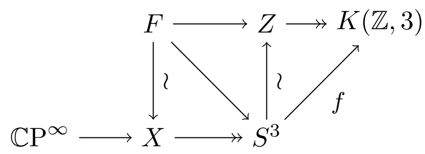


We may assume that $d(a_{k+1}) = a_k x$ and note that $a_k x = x a_k$.
We distinguish two cases.
If n is odd we compute by induction to $i + j$ that $a_i a_j = \binom{i+j}{i} a_{i+j}$. Hence $H^*(\Omega S^n, \mathbb{Z}) \simeq \Gamma_{\mathbb{Z}}[a_1]$, where the *divided polynomial algebra* $\Gamma_R[\alpha]$ is the quotient of the free R-algebra $R[\alpha_1, \alpha_2, \ldots]$ by the relations $\alpha_i \alpha_j = \binom{i+j}{i} \alpha_{i+j}$.
If n is even, then we compute $a_1^2 = 0$ and by induction on k we compute $a_1 a_{2k} = a_{2k+1}$ and $a_1 a_{2k+1} = 0$ and $a_2^k = k! a_{2k}$.
Now $H^*(\Omega S^n, \mathbb{Z}) \simeq \Lambda_{\mathbb{Z}}[a_1] \otimes \Gamma_{\mathbb{Z}}[a_2]$ where the *exterior algebra* $\Lambda_R[\alpha_1, \alpha_2, \ldots]$ is the free R-module with basis finite products $\alpha_{i_1} \cdots \alpha_{i_k}$ for $i_1 < \cdots < i_k$ where multiplication is defined as $\alpha_i \alpha_j = -\alpha_j \alpha_i$ and $\alpha_i^2 = 0$. △

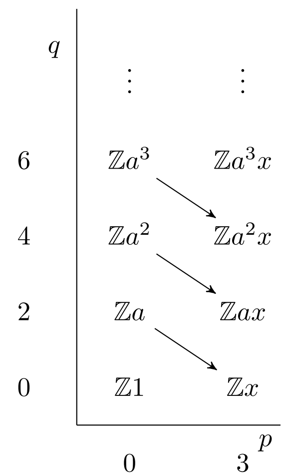
For the next example, we use the following.

Remark 3.5. We can factor any map $f : A \to B$ as a homotopy equivalence followed by a fibration: $A \xrightarrow{\sim} E_f \to B$. Here $E_f = \{(a, \gamma) \in A \times B^I \mid \gamma(0) = f(a)\}$. In HoTT we would have $E_f = \Sigma(x : A)\Sigma(y : B), f(a) = y$.

Example 3.6. In this example we will proof that the p-torsion subgroups of $\pi_i(S^3)$ is 0 for $i < 2p$ and $\mathbb{Z}_p$ for $i = 2p$. Start with a map $S^3 \to K(\mathbb{Z}, 3)$ inducing an isomorphism on π_3. Turn this into a fibration with fiber F. By the LES of homotopy groups of a fibration we get that F is 3-connected and $\pi_i(F) = \pi_i(S^3)$ for $i > 3$. Now convert the map $F \to S^3$ into a fibration. By the LES we see that the fiber is $K(\mathbb{Z}, 2) = \mathbb{C}\mathrm{P}^\infty$.



We now use the Serre Spectral Sequence of this last fibration. We know the homology groups of S^3 and $\mathbb{C}\mathrm{P}^\infty$, so we know the second page looks like this. Here the arrows are *not* all isomorphisms.



△

Since X is 3-connected, $d : \mathbb{Z}a \to \mathbb{Z}x$ must be an iso, so we may assume $da = x$. Then $d(a^n) = na^{n-1}x$. Now we know what groups survive until E_∞, to compute

$$H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}_n & \text{if } i = 2n+1 \\ 0 & \text{if } i = 2n, \end{cases} \qquad \text{hence} \qquad H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}_n & \text{if } i = 2n > 0 \\ 0 & \text{if } i = 2n-1. \end{cases}$$

The Hurewicz Theorem modulo $\mathcal{C}$ now implies that the first p-torsion in $\pi_*(X)$, hence also in $\pi_*(S^3)$ is a $\mathbb{Z}_p$ in π_{2p}. For $p = 2$ we get the stronger result that $\pi_4(S^3) = \mathbb{Z}^2$, hence also $\pi_{n+1}(S^n) = \mathbb{Z}_2$ for $n \geq 3$.