Carleson operators on doubling metric measure spaces

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ABSTRACT. We prove a new generalization of a theorem of Carleson, namely bounds for a generalized Carleson operator on doubling metric measure spaces. Additionally, we explicitly reduce Carleson's classical result on pointwise convergence of Fourier series to this new theorem. Both proofs are presented in great detail, suitable as a blueprint for computer verification using the current capabilities of the software package Lean. Note that even Carleson's classical result has not yet been computer-verified.

Contents

Introduction

In [Car66], L. Carleson addressed a classical question regarding the convergence of Fourier series of continuous functions by proving their pointwise convergence almost everywhere. Theorem 1.0.1 represents a version of this result.

Let f be a complex valued, 2π-periodic bounded Borel measurable function on the real line, and for an integer n, define the Fourier coefficient as

$$\widehat{f}_n := \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx. \tag{1.0.1}$$

Define the partial Fourier sum for $N \geq 0$ as

$$S_N f(x) := \sum_{n=-N}^{N} \widehat{f}_n e^{inx} . \tag{1.0.2}$$

THEOREM 1.0.1 (classical Carleson). *Let f be a 2π-periodic complex-valued continuous function on $\mathbb{R}$. For all $\epsilon > 0$, there exists a Borel set $E \subset [0, 2\pi]$ with Lebesgue measure $|E| \leq \epsilon$ and a positive integer N_0 such that for all $x \in [0, 2\pi] \smallsetminus E$ and all integers $N > N_0$, we have*

$$|f(x) - S_N f(x)| \leq \epsilon. \tag{1.0.3}$$

By applying this theorem with a sequence of $\epsilon_n := 2^{-n}\delta$ for $n \geq 1$ and taking the union of corresponding exceptional sets E_n, we see that outside a set of measure δ, the partial Fourier sums converge pointwise for $N \to \infty$. Applying this with a sequence of δ shrinking to zero and taking the intersection of the corresponding exceptional sets, which has measure zero, we see that the Fourier series converges outside a set of measure zero.

The purpose of this paper is twofold. On the one hand, it prepares computer verification of Theorem 1.0.1 by presenting a very detailed proof as a blueprint for coding in Lean. We pass through a bound for a generalization of the so-called Carleson operator to doubling metric measure spaces. This generalization is new, and proving these bounds constitutes the second purpose of this paper. This generalization incorporates several results from the recent literature, most prominently bounds for the polynomial Carleson operator of V. Lie [Lie20] as well as its generalization [Zor21]. A computer verification of our theorem will also entail a computer verification for the bulk of the work in these results.

We proceed to introduce the setup for our general theorem. We carry a multi purpose parameter, a natural number

$$a \geq 4 \tag{1.0.4}$$

in our notation that as it gets larger will allow more general applications but will worsen the constants in the estimates.

A doubling metric measure space (X, ρ, μ, a) is a complete and locally compact metric space (X, ρ) equipped with a σ-finite non-zero Radon–Borel measure μ that satisfies the doubling condition that for all $x \in X$ and all $R > 0$ we have

$$\mu(B(x, 2R)) \leq 2^a \mu(B(x, R)), \tag{1.0.5}$$

where we have denoted by $B(x, R)$ the open ball of radius R centred at x:

$$B(x, R) := \{y \in X : \rho(x, y) < R\}. \tag{1.0.6}$$

A collection Θ of real valued continuous functions on the doubling metric measure space (X, ρ, μ, a) is called compatible, if there is a point $o \in X$ where all the functions are equal to 0, and if there exists for each ball $B \subset X$ a metric d_B on Θ, such that the following five properties (1.0.7), (1.0.8), (1.0.9), (1.0.10), and (1.0.11) are satisfied. For every ball $B \subset X$

$$\sup_{x,y \in B} |\vartheta(x) - \vartheta(y) - \theta(x) + \theta(y)| \leq d_B(\vartheta, \theta). \tag{1.0.7}$$

For any two balls $B_1 = B(x_1, R)$, $B_2 = B(x_2, 2R)$ in X with $x_1 \in B_2$ and any $\vartheta, \theta \in \Theta$,

$$d_{B_2}(\vartheta, \theta) \leq 2^a d_{B_1}(\vartheta, \theta). \tag{1.0.8}$$

For any two balls B_1, B_2 in X with $B_1 \subset B_2$ and any $\vartheta, \theta \in \Theta$

$$d_{B_1}(\vartheta, \theta) \leq d_{B_2}(\vartheta, \theta) \tag{1.0.9}$$

and for any two balls $B_1 = B(x_1, R)$, $B_2 = B(x_2, 2^a R)$ with $B_1 \subset B_2$, and $\vartheta, \theta \in \Theta$,

$$2d_{B_1}(\vartheta, \theta) \leq d_{B_2}(\vartheta, \theta). \tag{1.0.10}$$

For every ball B in X and every d_B-ball $\tilde{B}$ of radius $2R$ in Θ, there is a collection $\mathcal{B}$ of at most 2^a many d_B-balls of radius R covering $\tilde{B}$, that is,

$$\tilde{B} \subset \bigcup \mathcal{B}. \tag{1.0.11}$$

Further, a compatible collection Θ is called cancellative, if for any ball B in X of radius R, any Lipschitz function $\varphi : X \to \mathbb{C}$ supported on B, and any $\vartheta, \theta \in \Theta$ we have

$$\left| \int_B e(\vartheta(x) - \theta(x))\varphi(x)d\mu(x) \right| \leq 2^a \mu(B) \|\varphi\|_{\mathrm{Lip}(B)} (1 + d_B(\vartheta, \theta))^{-\frac{1}{a}}, \tag{1.0.12}$$

where $\| \cdot \|_{\mathrm{Lip}(B)}$ denotes the inhomogeneous Lipschitz norm on B:

$$\|\varphi\|_{\mathrm{Lip}(B)} = \sup_{x \in B} |\varphi(x)| + R \sup_{x,y \in B, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\rho(x, y)}.$$

A one-sided Calderón–Zygmund kernel K on the doubling metric measure space (X, ρ, μ, a) is a measurable function

$$K : X \times X \to \mathbb{C} \tag{1.0.13}$$

such that for all $x, y', y \in X$ with $x \neq y$, we have

$$|K(x, y)| \leq \frac{2^{a^3}}{V(x, y)} \tag{1.0.14}$$

and if $2\rho(y, y') \leq \rho(x, y)$, then

$$|K(x, y) - K(x, y')| \leq \left(\frac{\rho(y, y')}{\rho(x, y)} \right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x, y)}, \tag{1.0.15}$$

where

$$V(x, y) := \mu(B(x, \rho(x, y))).$$

Define the maximally truncated non-tangential singular integral T_* associated with K by

$$T_* f(x) := \sup_{R_1 < R_2} \sup_{\rho(x, x') < R_1} \left| \int_{R_1 < \rho(x', y) < R_2} K(x', y) f(y) \, \mathrm{d}\mu(y) \right|. \tag{1.0.16}$$

We define the generalized Carleson operator T by

$$Tf(x) := \sup_{\vartheta \in \Theta} \sup_{0 < R_1 < R_2} \left| \int_{R_1 < \rho(x, y) < R_2} K(x, y) f(y) e(\vartheta(y)) \, \mathrm{d}\mu(y) \right|, \tag{1.0.17}$$

where $e(r) = e^{ir}$.

Our main result is the following restricted weak type estimate for T in the range $1 < q \leq 2$, which by interpolation techniques recovers L^q estimates for the open range $1 < q < 2$.

THEOREM 1.0.2 (metric space Carleson). *For all integers $a \geq 4$ and real numbers $1 < q \leq 2$ the following holds. Let (X, ρ, μ, a) be a doubling metric measure space. Let Θ be a cancellative compatible collection of functions and let K be a one-sided Calderón–Zygmund kernel on (X, ρ, μ, a). Assume that for every bounded measurable function g on X supported on a set of finite measure we have*

$$\|T_* g\|_2 \leq 2^{a^3} \|g\|_2, \tag{1.0.18}$$

where T_* is defined in (1.0.16). Then for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have, with T defined in (1.0.17),

$$\left| \int_G Tf \, \mathrm{d}\mu \right| \leq \frac{2^{450a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}. \tag{1.0.19}$$

For a Borel function $Q \to \Theta$, and $\vartheta \in \Theta$ and $x \in X$ define

$$R_Q(\vartheta, x) = \sup\{r > 0 : d_{B(x,r)}(\vartheta, Q(x)) < 1\} \tag{1.0.20}$$

and define further

$$T_Q^\vartheta f(x) := \sup_{R_1 > 0} \sup_{\rho(x, x') < R_1} \left| \int_{R_1 < \rho(x', y) < R_Q(\vartheta, x')} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| \tag{1.0.21}$$

Define further the linearized generalized Carleson operator T_Q by

$$T_Q f(x) := \sup_{0 < R_1 < R_2} \left| \int_{R_1 < \rho(x,y) < R_2} K(x,y) f(y) e(Q(x)(y)) \, d\mu(y) \right|, \qquad (1.0.22)$$

where again $e(r) = e^{ir}$.

THEOREM 1.0.3 (linearised metric Carleson). *For all integers $a \geq 4$ and real numbers $1 < q \leq 2$ the following holds. Let (X, ρ, μ, a) be a doubling metric measure space. Let Θ be a cancellative compatible collection of functions. Let $Q : X \to \Theta$ be a Borel function with finite range . Let K be a one-sided Calderón–Zygmund kernel on (X, ρ, μ, a). Assume that for every $\vartheta \in \Theta$ and every bounded measurable function g on X supported on a set of finite measure we have*

$$\|T_Q^\vartheta g\|_2 \leq 2^{a^3} \|g\|_2, \qquad (1.0.23)$$

where T_Q^ϑ is defined in (1.0.21). Then for all bounded Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have, with T_Q defined in (1.0.22),

$$\left| \int_G T_Q f \, d\mu \right| \leq \frac{2^{450a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}. \qquad (1.0.24)$$

We note that in Chapter **??** we formulate a variant of this theorem where Q has countable range.

In the one-dimensional Euclidean setting, with K representing the Hilbert kernel:

$$K(x,y) = (x-y)^{-1}$$

and Θ denoting the class of linear functions, the operator (1.0.17) is the classical Carleson operator, which plays a crucial role in proving the almost everywhere convergence of Fourier series [Car66], [Fef73], [LT00]. The supremum in R_1 and R_2 is often omitted in classical treatments, but considering the maximal truncations can easily be reduced to the case without these truncations.

By replacing Θ with the class of polynomials vanishing at 0 up to some fixed but arbitrary degree, we obtain the polynomial Carleson operator of Lie [Lie09] (quadratic case) and [Lie20]. The case of the class of polynomials with vanishing linear coefficient is simpler and was estimated in [SW01]. The polynomial Carleson operator was generalized to the high-dimensional Euclidean setting in [Zor21] for K being a Calderón-Zygmund kernel with some Hölder regularity.

Doubling metric measure spaces are instances of spaces of homogeneous type. Indeed, by changing from a quasi-metric to an equivalent metric, every space of homogeneous type can be viewed as a doubling metric measure space (cf. [MS79]). Spaces of homogeneous type were introduced by [CW71] as a natural setting for Calderón-Zygmund theory. We refer to the textbook [Ste93] for an account of these spaces.

Our concept of a compatible collection Θ as a natural class of phase functions on a doubling metric measure space does not appear in [Ste93] but is implicitly anticipated in [Zor21] and subsequent work of [Mna22], who proves a Carleson-type theorem for the Malmquist-Takenaka series, which leads to classes of phases related to Blaschke products. A generalization of (1.0.17) from the previously mentioned

Euclidean setting into the anisotropic setting that was suggested in [Zor21] is included in our theory. The polynomial Carleson operator also plays a role in the study of maximally modulated singular Radon transforms along the parabola, see [Ram21] and [Bec24].

For the proof of Theorem 1.0.2, we largely follow [Zor21], which in turn was inspired by [Lie20]. We make suitable modifications to adapt to our more general setting and have made a few technical improvements in the proof. In particular, in Chapter 2, we explicitly divide the main work of the proof into mutually independent sections 3, 4, 5, 6, 7, 8, and 9. Some of these sections follow a similar pattern, starting with a subsection dividing the proof into further mutually independent subsections. This modularization of our proof was strongly endorsed in personal communication by the author of [Zor21].

Proof of Metric Space Carleson, overview

This section organizes the proof of Theorem 1.0.2 into sections 3, 4, 5, 6, 7, 8, and 9. These sections are mutually independent except for referring to the statements formulated in the present section. Chapter 3 proves the main Theorem 1.0.2, while sections 4, 5, 6, 7, 8, and 9 each prove one proposition that is stated in the present section. The present section also introduces all definitions used across these sections.

Section 2.1 proves some auxiliary lemmas that are used in more than one of the sections 3-9.

Let a, q be given as in Theorem 1.0.2.

Define

$$D := 2^{100a^2} \, , \tag{2.0.1}$$

$$\kappa := 2^{-10a} \, , \tag{2.0.2}$$

and

$$Z := 2^{12a} \, . \tag{2.0.3}$$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the unique compactly supported, piece-wise linear, continuous function with corners precisely at $\frac{1}{4D}$, $\frac{1}{2D}$, $\frac{1}{4}$ and $\frac{1}{2}$ which satisfies

$$\sum_{s \in \mathbb{Z}} \psi(D^{-s}x) = 1 \tag{2.0.4}$$

for all $x > 0$. This function vanishes outside $[\frac{1}{4D}, \frac{1}{2}]$, is constant one on $[\frac{1}{2D}, \frac{1}{4}]$, and is Lipschitz with constant $4D$.

Let a doubling metric measure space (X, ρ, μ, a) be given. Let a cancellative compatible collection Θ of functions on X be given. Let $o \in X$ be a point such that $\vartheta(o) = 0$ for all $\vartheta \in \Theta$.

Let a Borel measurable function $Q : X \to \Theta$ with finite range be given. Let a one-sided Calderón–Zygmund kernel K on X be given so that for every $\vartheta \in \Theta$ the operator T_Q^ϑ defined in (1.0.21) satisfies (1.0.23).

For $s \in \mathbb{Z}$, we define

$$K_s(x, y) := K(x, y)\psi(D^{-s}\rho(x, y)) \, , \tag{2.0.5}$$

so that for each $x, y \in X$ with $x \neq y$ we have

$$K(x, y) = \sum_{s \in \mathbb{Z}} K_s(x, y).$$

In Chapter 3, we prove Theorem 1.0.2 and Theorem 1.0.3 from the finitary version, Proposition 2.0.1 below. Recall that a function from a measure space to a finite set is measurable if the pre-image of each of the elements in the range is measurable.

PROPOSITION 2.0.1 (finitary Carleson). *Let $\sigma_1, \sigma_2 \colon X \to \mathbb{Z}$ be measurable functions with finite range and $\sigma_1 \leq \sigma_2$. Let F, G be bounded Borel sets in X. Then there is a Borel set G' in X with $2\mu(G') \leq \mu(G)$ such that for all Borel functions $f \colon X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$.*

$$\int_{G \smallsetminus G'} \left| \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y) \right| \mathrm{d}\mu(x)$$

$$\leq \frac{2^{440a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}} . \tag{2.0.6}$$

Let measurable functions $\sigma_1 \leq \sigma_2 \colon X \to \mathbb{Z}$ with finite range be given. Let bounded Borel sets F, G in X be given. Let S be the smallest integer such that the ranges of σ_1 and σ_2 are contained in $[-S, S]$ and F and G are contained in the ball $B(o, \frac{1}{4} D^S)$.

In Chapter 4, we prove Proposition 2.0.1 using a bound for a dyadic model formulated in Proposition 2.0.2 below.

A grid structure $(\mathcal{D}, c, s)$ on X consists of a finite collection $\mathcal{D}$ of pairs (I, k) of Borel sets in X and integers $k \in [-S, S]$, the projection $s \colon \mathcal{D} \to [-S, S], (I, k) \mapsto k$ to the second component which is assumed to be surjective and called scale function, and a function $c \colon \mathcal{D} \to X$ called center function such that the five properties (2.0.7), (2.0.8), (2.0.9), (2.0.10), and (2.0.11) hold. We call the elements of $\mathcal{D}$ dyadic cubes. By abuse of notation, we will usually write just I for the cube (I, k), and we will write $I \subset J$ to mean that for two cubes $(I, k), (J, l) \in \mathcal{D}$ we have $I \subset J$ and $k \leq l$.

For each dyadic cube I and each $-S \leq k < s(I)$ we have

$$I \subset \bigcup_{J \in \mathcal{D}: s(J)=k} J . \tag{2.0.7}$$

Any two non-disjoint dyadic cubes I, J with $s(I) \leq s(J)$ satisfy

$$I \subset J . \tag{2.0.8}$$

There exists a $I_0 \in \mathcal{D}$ with $s(I_0) = S$ and $c(I_0) = o$ and for all cubes $J \in \mathcal{D}$, we have

$$J \subset I_0 . \tag{2.0.9}$$

For any dyadic cube I,

$$c(I) \in B(c(I), \frac{1}{4} D^{s(I)}) \subset I \subset B(c(I), 4 D^{s(I)}) . \tag{2.0.10}$$

For any dyadic cube I and any t with $t D^{s(I)} \geq D^{-S}$,

$$\mu(\{x \in I \ : \ \rho(x, X \smallsetminus I) \leq t D^{s(I)}\}) \leq 2 t^\kappa \mu(I) . \tag{2.0.11}$$

A tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$ for a given grid structure $(\mathcal{D}, c, s)$ is a finite set $\mathfrak{P}$ of elements called tiles with five maps

$$\mathcal{I} \colon \mathfrak{P} \to \mathcal{D}$$
$$\Omega \colon \mathfrak{P} \to \mathcal{P}(\Theta)$$
$$\mathcal{Q} \colon \mathfrak{P} \to Q(X)$$
$$\mathrm{c} \colon \mathfrak{P} \to X$$
$$\mathrm{s} \colon \mathfrak{P} \to \mathbb{Z}$$

with $\mathcal{I}$ surjective and $\mathcal{P}(\Theta)$ denoting the power set of Θ such that the five properties (2.0.13), (2.0.14), (2.0.15), (2.0.18), and (2.0.19) hold. For each dyadic cube I, the restriction of the map Ω to the set

$$\mathfrak{P}(I) = \{\mathfrak{p} : \mathcal{I}(\mathfrak{p}) = I\} \tag{2.0.12}$$

is injective and we have the disjoint covering property (we use the union symbol with dot on top to denote a disjoint union)

$$Q(X) \subset \dot{\bigcup}_{\mathfrak{p} \in \mathfrak{P}(I)} \Omega(\mathfrak{p}). \tag{2.0.13}$$

For any tiles $\mathfrak{p}, \mathfrak{q}$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{q})$ and $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) \neq \varnothing$ we have

$$\Omega(\mathfrak{q}) \subset \Omega(\mathfrak{p}). \tag{2.0.14}$$

For each tile $\mathfrak{p}$,

$$\mathcal{Q}(\mathfrak{p}) \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \subset \Omega(\mathfrak{p}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 1), \tag{2.0.15}$$

where

$$B_{\mathfrak{p}}(\vartheta, R) := \{\theta \in \Theta \,:\, d_{\mathfrak{p}}(\vartheta, \theta) < R\}, \tag{2.0.16}$$

and

$$d_{\mathfrak{p}} := d_{B(\mathrm{c}(\mathfrak{p}), \frac{1}{4}D^{\mathrm{s}(\mathfrak{p})})}. \tag{2.0.17}$$

We have for each tile $\mathfrak{p}$

$$\mathrm{c}(\mathfrak{p}) = c(\mathcal{I}(\mathfrak{p})), \tag{2.0.18}$$
$$\mathrm{s}(\mathfrak{p}) = s(\mathcal{I}(\mathfrak{p})). \tag{2.0.19}$$

PROPOSITION 2.0.2 (discrete Carleson). *Let $(\mathcal{D}, c, s)$ be a grid structure and*

$$(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$$

a tile structure for this grid structure. Define for $\mathfrak{p} \in \mathfrak{P}$

$$E(\mathfrak{p}) = \{x \in \mathcal{I}(\mathfrak{p}) : Q(x) \in \Omega(\mathfrak{p}), \sigma_1(x) \leq \mathrm{s}(\mathfrak{p}) \leq \sigma_2(x)\} \tag{2.0.20}$$

and

$$T_{\mathfrak{p}}f(x) = \mathbf{1}_{E(\mathfrak{p})}(x) \int K_{\mathrm{s}(\mathfrak{p})}(x, y) f(y) e(Q(x)(y) - Q(x)(x)) \, d\mu(y). \tag{2.0.21}$$

Then there exists a Borel set G' with $2\mu(G') \leq \mu(G)$ such that for all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ we have

$$\int_{G \setminus G'} \left| \sum_{\mathfrak{p} \in \mathfrak{P}} T_{\mathfrak{p}} f(x) \right| \, d\mu(x) \leq \frac{2^{440a^3}}{(q-1)^4} \mu(G)^{1 - \frac{1}{q}} \mu(F)^{\frac{1}{q}}. \tag{2.0.22}$$

The proof of Proposition 2.0.2 is done in Chapter 5 by a reduction to two further propositions that we state below.

Fix a grid structure $(\mathcal{D}, c, s)$ and a tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, c, s)$ for this grid structure.

We define the relation

$$\mathfrak{p} \leq \mathfrak{p}' \qquad (2.0.23)$$

on $\mathfrak{P} \times \mathfrak{P}$ meaning $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$ and $\Omega(\mathfrak{p}') \subset \Omega(\mathfrak{p})$. We further define for $\lambda, \lambda' > 0$ the relation

$$\lambda \mathfrak{p} \lesssim \lambda' \mathfrak{p}' \qquad (2.0.24)$$

on $\mathfrak{P} \times \mathfrak{P}$ meaning $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$ and

$$B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \lambda') \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \lambda). \qquad (2.0.25)$$

Define for a tile $\mathfrak{p}$ and $\lambda > 0$

$$E_1(\mathfrak{p}) := \{x \in \mathcal{I}(\mathfrak{p}) \cap G : Q(x) \in \Omega(\mathfrak{p})\}, \qquad (2.0.26)$$

$$E_2(\lambda, \mathfrak{p}) := \{x \in \mathcal{I}(\mathfrak{p}) \cap G : Q(x) \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \lambda)\}. \qquad (2.0.27)$$

Given a subset $\mathfrak{P}'$ of $\mathfrak{P}$, we define $\mathfrak{P}(\mathfrak{P}')$ to be the set of all $\mathfrak{p} \in \mathfrak{P}$ such that there exist $\mathfrak{p}' \in \mathfrak{P}'$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$. Define the densities

$$\mathrm{dens}_1(\mathfrak{P}') := \sup_{\mathfrak{p}' \in \mathfrak{P}'} \sup_{\lambda \geq 2} \lambda^{-a} \sup_{\mathfrak{p} \in \mathfrak{P}(\mathfrak{P}'), \lambda \mathfrak{p}' \lesssim \lambda \mathfrak{p}} \frac{\mu(E_2(\lambda, \mathfrak{p}))}{\mu(\mathcal{I}(\mathfrak{p}))}, \qquad (2.0.28)$$

$$\mathrm{dens}_2(\mathfrak{P}') := \sup_{\mathfrak{p}' \in \mathfrak{P}'} \sup_{r \geq 4D^{s(\mathfrak{p})}} \frac{\mu(F \cap B(c(\mathfrak{p}), r))}{\mu(B(c(\mathfrak{p}), r))}. \qquad (2.0.29)$$

An antichain is a subset $\mathfrak{A}$ of $\mathfrak{P}$ such that for any distinct $\mathfrak{p}, \mathfrak{q} \in \mathfrak{A}$ we do not have have $\mathfrak{p} \leq \mathfrak{q}$.

The following proposition is proved in Chapter 6.

PROPOSITION 2.0.3 (antichain operator). *For any antichain $\mathfrak{A}$ and for all $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ and all $g : X \to \mathbb{C}$ with $|g| \leq \mathbf{1}_G$*

$$|\int \overline{g(x)} \sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x) \, d\mu(x)| \qquad (2.0.30)$$

$$\leq \frac{2^{150a^3}}{q-1} \mathrm{dens}_1(\mathfrak{A})^{\frac{q-1}{8a^4}} \mathrm{dens}_2(\mathfrak{A})^{\frac{1}{q}-\frac{1}{2}} \|f\|_2 \|g\|_2. \qquad (2.0.31)$$

Let $n \geq 0$. An n-forest is a pair $(\mathfrak{U}, \mathfrak{T})$ where $\mathfrak{U}$ is a subset of $\mathfrak{P}$ and $\mathfrak{T}$ is a map assigning to each $\mathfrak{u} \in \mathfrak{U}$ a nonempty set $\mathfrak{T}(\mathfrak{u}) \subset \mathfrak{P}$ called tree such that the following properties (2.0.32), (2.0.33), (2.0.34), (2.0.35), (2.0.36), and (2.0.37) hold.

For each $\mathfrak{u} \in \mathfrak{U}$ and each $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ we have $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u})$ and

$$4\mathfrak{p} \lesssim 1\mathfrak{u}. \qquad (2.0.32)$$

For each $\mathfrak{u} \in \mathfrak{U}$ and each $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u})$ and $\mathfrak{p}' \in \mathfrak{P}$ we have

$$\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u}), \mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}'' \implies \mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}). \qquad (2.0.33)$$

We have

$$\|\sum_{\mathfrak{u} \in \mathfrak{U}} \mathbf{1}_{\mathcal{I}(\mathfrak{u})}\|_\infty \leq 2^n. \qquad (2.0.34)$$

We have for every $\mathfrak{u} \in \mathfrak{U}$

$$\mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \leq 2^{4a+1-n} \, . \tag{2.0.35}$$

We have for $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}$ with $\mathfrak{u} \neq \mathfrak{u}'$ and $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}')$ with $\mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u})$ that

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u})) > 2^{Z(n+1)} \, . \tag{2.0.36}$$

We have for every $\mathfrak{u} \in \mathfrak{U}$ and $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ that

$$B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})}) \subset \mathcal{J}(\mathfrak{u}). \tag{2.0.37}$$

The following proposition is proved in Chapter 7.

PROPOSITION 2.0.4 (forest operator). *For any $n \geq 0$ and any n-forest $(\mathfrak{U}, \mathfrak{T})$ we have for all $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ and all bounded g with bounded support*

$$|\int \overline{g(x)} \sum_{\mathfrak{u} \in \mathfrak{U}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(x) \, \mathrm{d}\mu(x)|$$

$$\leq 2^{432a^3} 2^{-\frac{q-1}{q}n} \, \mathrm{dens}_2\left(\bigcup_{\mathfrak{u} \in \mathfrak{U}} \mathfrak{T}(\mathfrak{u})\right)^{\frac{1}{q}-\frac{1}{2}} \|f\|_2 \|g\|_2 \, .$$

Theorem 1.0.2 is formulated at the level of generality for general kernels satisfying the mere Hölder regularity condition (1.0.15). On the other hand, the cancellative condition (1.0.12) is a testing condition against more regular, namely Lipschitz functions. To bridge the gap, we follow [Zor21] to observe a variant of (1.0.12) that we formulate in the following proposition proved in Chapter 8.

Define

$$\tau := \frac{1}{a} \, . \tag{2.0.38}$$

Define for any open ball B of radius R in X the L^∞-normalized τ-Hölder norm by

$$\|\varphi\|_{C^\tau(B)} = \sup_{x \in B} |\varphi(x)| + R^\tau \sup_{x,y \in B, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\rho(x,y)^\tau} \, . \tag{2.0.39}$$

PROPOSITION 2.0.5 (Holder van der Corput). *Let $z \in X$ and $R > 0$ and set $B = B(z, R)$. Let $\varphi : X \to \mathbb{C}$ by supported on B and satisfy $\|\varphi\|_{C^\tau(B)} < \infty$. Let $\vartheta, \theta \in \Theta$. Then*

$$|\int e(\vartheta(x) - \theta(x))\varphi(x)dx| \leq 2^{8a}\mu(B)\|\varphi\|_{C^\tau(B)}(1 + d_B(\vartheta, \theta))^{-\frac{1}{2a^2+a^3}} \, . \tag{2.0.40}$$

We further formulate a classical Vitali covering result and maximal function estimate that we need throughout several sections. This following proposition will typically be applied to the absolute value of a complex valued function and be proved in Chapter 9. By a ball B we mean a set $B(x, r)$ with $x \in X$ and $r > 0$ as defined in (1.0.6). For a finite collection $\mathcal{B}$ of balls in X and $1 \leq p < \infty$ define the measurable function $M_{\mathcal{B},p}u$ on X by

$$M_{\mathcal{B},p}u(x) := \left(\sup_{B \in \mathcal{B}} \frac{\mathbf{1}_B(x)}{\mu(B)} \int_B |u(y)|^p \, d\mu(y)\right)^{\frac{1}{p}} \, . \tag{2.0.41}$$

Define further $M_{\mathcal{B}} := M_{\mathcal{B},1}$.

PROPOSITION 2.0.6 (Hardy–Littlewood). *Let $\mathcal{B}$ be a finite collection of balls in X. If for some $\lambda > 0$ and some measurable function $u : X \to [0, \infty)$ we have*

$$\int_B u(x)\, d\mu(x) \geq \lambda \mu(B) \tag{2.0.42}$$

for each $B \in \mathcal{B}$, then

$$\lambda \mu(\bigcup \mathcal{B}) \leq 2^{2a} \int_X u(x)\, d\mu(x). \tag{2.0.43}$$

For every measurable function v and $1 \leq p_1 < p_2$ we have

$$\|M_{\mathcal{B},p_1} v\|_{p_2} \leq 2^{2a} \frac{p_2}{p_2 - p_1} \|v\|_{p_2}. \tag{2.0.44}$$

Moreover, given any measurable bounded function $w : X \to \mathbb{C}$ there exists a measurable function $Mw : X \to [0, \infty)$ such that the following (2.0.45) and (2.0.46) hold. For each ball $B \subset X$ and each $x \in B$

$$\frac{1}{\mu(B)} \int_B |w(y)|\, \mathrm{d}\mu(y) \leq Mw(x) \tag{2.0.45}$$

and for all $1 \leq p_1 < p_2 \leq \infty$

$$\|M(w^{p_1})^{\frac{1}{p_1}}\|_{p_2} \leq 2^{4a} \frac{p_2}{p_2 - p_1} \|w\|_{p_2}. \tag{2.0.46}$$

This completes the overview of the proof of Theorem 1.0.2.

2.1. Auxiliary lemmas

We close this section by recording some auxiliary lemmas about the objects defined in Chapter 2, which will be used in multiple sections to follow.

First, we record an estimate for the metrical entropy numbers of balls in the space Θ equipped with any of the metrics d_B, following from the doubling property (1.0.11).

LEMMA 2.1.1 (ball metric entropy). *Let $B' \subset X$ be a ball. Let $r > 0$, $\vartheta \in \Theta$ and $k \in \mathbb{N}$. Suppose that $\mathcal{Z} \subset B_{B'}(\vartheta, r2^k)$ satisfies that $\{B_{B'}(z, r) \mid z \in \mathcal{Z}\}$ is a collection of pairwise disjoint sets. Then*

$$|\mathcal{Z}| \leq 2^{ka}.$$

PROOF. By applying property (1.0.11) k times, we obtain a collection $\mathcal{Z}' \subset \Theta$ with $|\mathcal{Z}'| = 2^{ka}$ and

$$B_{B'}(\vartheta, r2^k) \subset \bigcup_{z' \in \mathcal{Z}'} B_{B'}(z', \frac{r}{2}).$$

Then each $z \in \mathcal{Z}$ is contained in one of the balls $B(z', \frac{r}{2})$, but by the separation assumption no such ball contains more than one element of $\mathcal{Z}$. Thus $|\mathcal{Z}| \leq |\mathcal{Z}'| = 2^{ka}$. $\square$

The next lemma concerns monotonicity of the metrics $d_{B(c(I), \frac{1}{4} D^{s(I)})}$ with respect to inclusion of cubes I in a grid.

LEMMA 2.1.2 (monotone cube metrics). *Let $(\mathcal{D}, c, s)$ be a grid structure. Denote for cubes $I \in \mathcal{D}$*

$$I^\circ := B(c(I), \frac{1}{4} D^{s(I)}) \, .$$

Let $I, J \in \mathcal{D}$ with $I \subset J$. Then for all $\vartheta, \theta \in \Theta$ we have

$$d_{I^\circ}(\vartheta, \theta) \le d_{J^\circ}(\vartheta, \theta) \, ,$$

and if $I \ne J$ then we have

$$d_{I^\circ}(\vartheta, \theta) \le 2^{-95a} d_{J^\circ}(\vartheta, \theta) \, .$$

PROOF. If $s(I) \ge s(J)$ then (2.0.8) and the assumption $I \subset J$ imply $I = J$. Then the lemma holds by reflexivity.

If $s(J) \ge s(I) + 1$, then using the monotonicity property (1.0.9), (2.0.1) and (1.0.10), we get

$$d_{I^\circ}(\vartheta, \theta) \le d_{B(c(I), 4D^{s(I)})}(\vartheta, \theta) \le 2^{-100a} d_{B(c(I), 4D^{s(J)})}(\vartheta, \theta) \, . \tag{2.1.1}$$

Using (2.0.10), together with the inclusion $I \subset J$, we obtain

$$c(I) \in I \subset J \subset B(c(J), 4D^{s(J)})$$

and consequently by the triangle inequality

$$B(c(I), 4D^{s(J)}) \subset B(c(J), 8D^{s(J)}) \, .$$

Using this together with the monotonicity property (1.0.9) and (1.0.8) in (2.1.1), we obtain

$$\begin{aligned}
d_{I^\circ}(\vartheta, \theta) &\le 2^{-100a} d_{B(c(J), 8D^{s(J)})}(\vartheta, \theta) \\
&\le 2^{-100a+5a} d_{B(c(J), \frac{1}{4} D^{s(J)})}(\vartheta, \theta) \\
&= 2^{-95a} d_{J^\circ}(\vartheta, \theta) \, .
\end{aligned}$$

This proves the second inequality claimed in the Lemma, from which the first follows since $a \ge 4$ and hence $2^{-95a} \le 1$. $\qquad \square$

We also record the following basic estimates for the kernels K_s.

LEMMA 2.1.3 (kernel summand). *Let $-S \le s \le S$ and $x, y, y' \in X$. If $K_s(x, y) \ne 0$, then we have*

$$\frac{1}{4} D^{s-1} \le \rho(x, y) \le \frac{1}{2} D^s \, . \tag{2.1.2}$$

We have

$$|K_s(x, y)| \le \frac{2^{102a^3}}{\mu(B(x, D^s))} \tag{2.1.3}$$

and

$$|K_s(x, y) - K_s(x, y')| \le \frac{2^{150a^3}}{\mu(B(x, D^s))} \left(\frac{\rho(y, y')}{D^s} \right)^{\frac{1}{a}} \, . \tag{2.1.4}$$

PROOF. By Definition (2.0.5), the function K_s is the product of K with a function which is supported in the set of all x, y satisfying (2.1.2). This proves (2.1.2).

Using (1.0.14) and the lower bound in (2.1.2) we obtain

$$|K_s(x,y)| \leq |K(x,y)| \leq \frac{2^{a^3}}{\mu(B(x,\frac{1}{4}D^{s-1}))} \tag{2.1.5}$$

Using $D = 2^{100a^2}$ and the doubling property (1.0.5) $2 + 100a^2$ times estimates the last display by

$$\leq \frac{2^{2a+101a^3}}{\mu(B(x,D^s))}. \tag{2.1.6}$$

Using $a \geq 4$ proves (2.1.3).

To prove (2.1.4) when $2\rho(y,y') > \rho(x,y)$, use the lower bound in (2.1.2), $2\rho(y,y') > \frac{1}{4}D^{s-1}$. Then (2.1.4) follows from the triangle inequality, (2.1.3) and $a \geq 4$.

If $2\rho(y,y') \leq \rho(x,y)$, we rewrite $|K_s(x,y) - K_s(x,y')|$ as

$$|(K(x,y) - K(x,y'))\psi(D^{-s}\rho(x,y)) + K(x,y)(\psi(D^{-s}\rho(x,y)) - \psi(D^{-s}\rho(x,y')))|. \tag{2.1.7}$$

An upper bound for $|K(x,y) - K(x,y')|$ is obtained similarly to the proof of (2.1.3), using (1.0.15) and the lower bound in (2.1.2)

$$|K(x,y) - K(x,y')| \leq \frac{2^{a^3}}{\mu(B(x,\frac{1}{4}D^{s-1}))} \left(\frac{\rho(y,y')}{\frac{1}{4}D^{s-1}}\right)^{\frac{1}{a}}. \tag{2.1.8}$$

As above, this is estimated by

$$\leq \frac{4D2^{2a+101a^3}}{\mu(B(x,D^s))} \left(\frac{\rho(y,y')}{D^s}\right)^{\frac{1}{a}} = \frac{2^{2+2a+100a^2+101a^3}}{\mu(B(x,D^s))} \left(\frac{\rho(y,y')}{D^s}\right)^{\frac{1}{a}}. \tag{2.1.9}$$

We have the trivial bound $|\psi(D^{-s}\rho(x,y))| \leq 1$, and (2.1.6) provides a bound for $|K(x,y)|$. Finally, we show that

$$|\psi(D^{-s}\rho(x,y)) - \psi(D^{-s}\rho(x,y'))| \leq 4D\left(\frac{\rho(y,y')}{D^s}\right)^{\frac{1}{a}} \tag{2.1.10}$$

by considering separately the cases $\rho(y,y')/D^s \geq 1$ and $\rho(y,y')/D^s < 1$. In the former case, the inequality is trivial; in the latter case, it follows from the fact that ψ is Lipschitz with constant $4D$.

Combining the above bounds and using $a \geq 4$ proves (2.1.4) in the case $2\rho(y,y') \leq \rho(x,y)$.   □

CHAPTER 3

Proof of Metric Space Carleson

This section is currently under construction, to incoroporate the proof of Theorem 1.0.3

Let Borel sets F, G in X be given. Let a Borel function $f : X \to \mathbb{C}$ with $f \leq \mathbf{1}_F$ be given. We have that

$$X = \bigcup_{R>0} B(o, R), \tag{3.0.1}$$

because every point of X has finite distance from o.

LEMMA 3.0.1 (R truncation). *For all integers $R > 0$*

$$\int \mathbf{1}_{G \cap B(o,R)} \sup_{1/R < R_1 < R_2 < R} \sup_{\vartheta \in \Theta} \left| T_{R_1, R_2, \vartheta} f(x) \right| d\mu(x)$$

$$\leq \frac{2^{450a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}, \tag{3.0.2}$$

where

$$T_{R_1, R_2, \vartheta} f(x) = \int_{R_1 < \rho(x,y) < R_2} K(x,y) f(y) e(\vartheta(y)) \, d\mu(y). \tag{3.0.3}$$

We first show how Lemma 3.0.1 implies Theorem 1.0.2. As R tends to ∞, the integrand of the left-hand side of (3.0.2) grows monotonically toward the integrand of the left-hand side of (1.0.19) for all x. By Lebesgue's monotone convergence theorem, the left-hand side of (3.0.2) converges to the left-hand side of (1.0.19). This verifies Theorem 1.0.2.

It remains to prove Lemma 3.0.1. Fix an integer $R > 0$. By replacing G with $G \cap B(o, R)$ if necessary, it suffices to show (3.0.2) under the assumption that G is contained in $B(o, R)$. We make this assumption. For every $x \in G$, the domain of integration in (3.0.3) is contained in $B(o, 2R)$. By replacing F with $F \cap B(o, 2R)$ if necessary, and correspondingly restricting f to $B(o, 2R)$, it suffices to show (3.0.2) under the assumption that F is contained in $B(o, 2R)$. We make this assumption.

Using the definition (2.0.5) of K_s and the partition of unity (2.0.4), we express (3.0.3) as the sum of

$$T_{1, s_1, s_2, \vartheta} f(x) := \sum_{s_1 \leq s \leq s_2} \int K_s(x,y) f(y) e(\vartheta(y)) \, d\mu(y) \tag{3.0.4}$$

and

$$\sum_{s = s_1 - 2, s_1 - 1, s_2 + 1, s_2 + 2} \int_{R_1 < \rho(x,y) < R_2} K_s(x,y) f(y) e(\vartheta(y)) \, d\mu(y), \tag{3.0.5}$$

where s_1 is the smallest integer such that $D^{s_1-2}R_2 > \frac{1}{4D}$ and s_2 is the largest integer such that $D^{s_2+2}R_1 < \frac{1}{2}$. We restrict the summation index s by excluding summands with $s < s_1 - 2$ or $s > s_2 + 2$ because for these summands, the function K_s vanishes on the domain of integration. We also omit the restriction in the integral for the summands in (3.0.4) because in these summands, the support of K_s is contained in the set described by this restriction.

We apply the triangle inequality and estimate the versions of (3.0.2) separately with $T_{R_1,R_2,\vartheta}$ replaced by (3.0.4) and by each summand of (3.0.5). To handle the case (3.0.4), we employ the following lemma. Here, we utilize the fact that if $\frac{1}{R} \leq R_1 \leq R_2 \leq R$, then s_1 and s_2 as in (3.0.4) are in an interval $[-S, S]$ for some sufficiently large S depending on R.

LEMMA 3.0.2 (S truncation). *For all integers $S > 0$*

$$\int \mathbf{1}_G(x) \max_{-S < s_1 \leq s_2 < S} \sup_{\vartheta \in \Theta} \left| T_{1,s_1,s_2,\vartheta} f(x) \right| d\mu(x)$$

$$\leq \frac{2^{446a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}, \tag{3.0.6}$$

where $T_{1,s_1,s_2,\vartheta}$ is defined in (3.0.4).

To reduce Lemma 3.0.1 to Lemma 3.0.2, we need estimates for the summands in (3.0.5). Using Lemma 2.1.3, we obtain for arbitrary s the inequality

$$\left| \int_{R_1 < \rho(x,y) < R_2} K_s(x,y) f(y) e(\vartheta(y)) \, d\mu(y) \right|$$

$$\leq \frac{2^{102a^3}}{\mu(B(x,D^s))} \int_{B(x,D^s)} \mathbf{1}_F(y) \, d\mu(y) \leq 2^{102a^3} M\mathbf{1}_F(x), \quad (3.0.7)$$

where $M\mathbf{1}_F$ is as defined in Proposition 2.0.6. Now, the left-hand side of (3.0.2), with $T_{R_1,R_2,\vartheta}$ replaced by a summand of (3.0.5), can be estimated using Hölder's inequality and Proposition 2.0.6 by

$$2^{102a^3} \int \mathbf{1}_G(x) M\mathbf{1}_F(x) \, d\mu(x) \leq \frac{2^{102a^3+4a}q}{q-1} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}} .$$

Applying the triangle inequality to estimate the left-hand side of (3.0.2) by contributions from the summands in (3.0.4) and (3.0.5), using Lemma 3.0.2 to control the first term, and the above to estimate the contribution from the four summands in (3.0.5), combined with $a \geq 4$ and $q < 2$, completes the reduction of Lemma 3.0.1 to Lemma 3.0.2.

It remains to prove Lemma 3.0.2. Fix $S > 0$.

LEMMA 3.0.3 (finitary S truncation). *For all finite sets $\widetilde{\Theta} \subset \Theta$*

$$\int \mathbf{1}_G(x) \max_{-S < s_1 \leq s_2 < S} \sup_{\vartheta \in \widetilde{\Theta}} \left| T_{1,s_1,s_2,\vartheta} f(x) \right| d\mu(x)$$

$$\leq \frac{2^{445a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}} . \tag{3.0.8}$$

We reduce Lemma 3.0.2 to Lemma 3.0.3. By the Lebesgue monotone convergence theorem, applied to an increasing sequence of finite sets $\tilde{\Theta}$, inequality (3.0.8) continues to hold for countable $\tilde{\Theta}$.

Let $\epsilon = \frac{1}{2S+1}$. Pick some $\vartheta_0 \in \Theta$. For $k \geq 0$, let the set $\tilde{\Theta}_k$ be a subset of $B_{B(o,2R)}(\vartheta_0, k)$ of maximal size, such that for all $\vartheta, \theta \in \tilde{\Theta}_k$, it holds that $d_{B(o,2R)}(\vartheta, \theta) \geq \epsilon$. Such a set exists, since by Lemma 2.1.1 there exists an upper bound for the size of such subsets in $B_{B(o,2R)}(\vartheta_0, k)$. Define

$$\tilde{\Theta} := \bigcup_{k \in \mathbb{N}} \tilde{\Theta}_k \,.$$

Then the set $\tilde{\Theta}$ is at most countable, and it has the property that for any $\theta \in \Theta$, there exists $\vartheta \in \tilde{\Theta}$ with

$$d_{B(o,2R)}(\theta, \vartheta) < \epsilon \,.$$

For every $\vartheta \in \Theta$, we have

$$\left| T_{1,s_1,s_2,\vartheta} f(x) \right| = \left| \sum_{s_1 \leq s \leq s_2} \int K_s(x,y) f(y) e(\vartheta(y) - \vartheta(x)) \, d\mu(y) \right| \tag{3.0.9}$$

Moreover, there is a $\tilde{\vartheta} \in \tilde{\Theta}$ with $d_{B(o,2R)}(\vartheta, \tilde{\vartheta}) \leq \epsilon$. Hence,

$$\left| T_{1,s_1,s_2,\vartheta} f(x) \right| - \left| T_{1,s_1,s_2,\tilde{\vartheta}} f(x) \right|$$

$$\leq \sum_{s_1 \leq s \leq s_2} \int |K_s(x,y)| \mathbf{1}_F(y) |e(\vartheta(y) - \vartheta(x)) - e(\tilde{\vartheta}(y) - \tilde{\vartheta}(x))| \, d\mu(y)$$

$$\leq \sum_{s_1 \leq s \leq s_2} \epsilon \int |K_s(x,y)| \mathbf{1}_F(y) \, d\mu(y)$$

Using Lemma 2.1.3, we can estimate the above expression by

$$\sum_{s_1 \leq s \leq s_2} \frac{2^{102a^3}}{\mu(B(x, D^s))} \epsilon \int_{B(x, D^s)} \mathbf{1}_F(y) \, d\mu(y)$$

$$\leq (2S+1) \epsilon 2^{102a^3} M \mathbf{1}_F(x) \leq 2^{102a^3} M \mathbf{1}_F(x)$$

We estimate the left-hand-side of (3.0.6) by the sum of left-hand-side of (3.0.8) and

$$\int \mathbf{1}_G(x) \max_{-S < s_1 \leq s_2 < S} \sup_{\vartheta \in \Theta} \inf_{\tilde{\vartheta} \in \tilde{\Theta}} (|T_{1,s_1,s_2,\vartheta} f(x)| - |T_{1,s_1,s_2,\tilde{\vartheta}} f(x)|) \, d\mu(x) \,,$$

which, as we have just shown, is estimated by

$$2^{102a^3} \int \mathbf{1}_G(x) M \mathbf{1}_F(x) \, d\mu(x) \,.$$

By Hölder's inequality and Proposition 2.0.6 (more precisely, (2.0.46) with $p = q$), the above is no greater than

$$\frac{2^{102a^3 + 4a} q}{q - 1} \mu(G)^{1 - \frac{1}{q}} \mu(F)^{\frac{1}{q}} \,.$$

Combining this with Lemma 3.0.3 and the fact that

$$\frac{2^{102a^3+4a}q}{q-1} \leq \frac{2^{445a^3}}{(q-1)^6}$$

proves Lemma 3.0.2.

It remains to prove Lemma 3.0.3. Fix a finite set $\tilde{\Theta}$.

LEMMA 3.0.4 (linearized truncation). *Let $\sigma_1, \sigma_2 \colon X \to \mathbb{Z}$ be measurable functions with finite range $[-S, S]$ and $\sigma_1 \leq \sigma_2$. Let $Q \colon X \to \tilde{\Theta}$ be a measurable function. Then we have*

$$\int \mathbf{1}_G(x) \left| T_{2,\sigma_1,\sigma_2,Q} f(x) \right| d\mu(x) \leq \frac{2^{445a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}, \qquad (3.0.10)$$

with

$$T_{2,\sigma_1,\sigma_2,Q} f(x) = \sum_{\sigma_1(x) \leq s \leq \sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y) - Q(x)(x)) \, d\mu(y). \quad (3.0.11)$$

We reduce Lemma 3.0.3 to Lemma 3.0.4. For each x, let $\sigma_1(x)$ be the minimal element $s' \in [-S, S]$ such that

$$\max_{s' \leq s_2 < S} \max_{\vartheta \in \tilde{\Theta}} |T_{1,s',s_2,\vartheta} f(x)| = \max_{-S < s_1 \leq s_2 < S} \max_{\vartheta \in \tilde{\Theta}} |T_{1,s_1,s_2,\vartheta} f(x)| := T_{1,x}.$$

Similarly, let $\sigma_2(x)$ be the minimal element $s'' \in [-S, S]$ such that

$$\max_{\vartheta \in \tilde{\Theta}} |T_{1,\sigma_1(x),s'',\vartheta} f(x)| = T_{1,x} .$$

Finally, choose a total order of the finite set $\tilde{\Theta}$ and let $Q(x)$ be the minimal element ϑ with respect to this order such that

$$|T_{1,\sigma_1(x),\sigma_2(x),\vartheta} f(x)| = T_{1,x} .$$

With these choices, and noting that

$$T_{1,\sigma_1(x),\sigma_2(x),Q(x)} f(x) = T_{2,\sigma_1,\sigma_2,Q} f(x),$$

we conclude that the left-hand side of (3.0.8) and (3.0.10) are equal. Thus, Lemma 3.0.3 follows from Lemma 3.0.4.

It remains to prove Lemma 3.0.4. Fix σ_1, σ_2, and Q as in the lemma. Applying Proposition 2.0.1 recursively, we obtain a sequence of sets G_n with $G_0 = G$ and, for each $n \geq 0$, $\mu(G_n) \leq 2^{-n}\mu(G)$ and

$$\int \mathbf{1}_{G_n \setminus G_{n+1}}(x) \left| T_{2,\sigma_1,\sigma_2,Q} f(x) \right| d\mu(x)$$

$$\leq \frac{2^{440a^3}}{(q-1)^5} \mu(G_n)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}, \qquad (3.0.12)$$

Adding the first n of these inequalities, we obtain by bounding a geometric series

$$\int \mathbf{1}_{G \setminus G_n}(x) \left| T_{2,\sigma_1,\sigma_2,Q} f(x) \right| d\mu(x) \leq \frac{2^{445a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}. \qquad (3.0.13)$$

As the integrand is non-negative and non-decreasing in n, we obtain by the monotone convergence theorem

$$\int \mathbf{1}_G(x) \left| T_{2,\sigma_1,\sigma_2,Q} f(x) \right| d\mu(x) \leq \frac{2^{445a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}. \tag{3.0.14}$$

This completes the proof of Lemma 3.0.4 and thus Theorem 1.0.2.

CHAPTER 4

Proof of Finitary Carleson

To prove Proposition 2.0.1, we already fixed in Chapter 2 measurable functions σ_1, σ_2, Q and Borel sets F, G. We have also defined S to be the smallest integer such that the ranges of σ_1 and σ_2 are contained in $[-S, S]$ and F and G are contained in the ball $B(o, \frac{1}{4}D^S)$.

The proof of the next lemma is done in Section 4.1, following the construction of dyadic cubes in [Chr90, §3].

LEMMA 4.0.1 (grid existence). *There exists a grid structure $(\mathcal{D}, c, s)$.*

The next lemma, which we prove in Section 4.2, should be compared with the construction in [Zor21, Lemma 2.12].

LEMMA 4.0.2 (tile structure). *For a given grid structure $(\mathcal{D}, c, s)$, there exists a tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, c, s)$.*

Choose a grid structure $(\mathcal{D}, c, s)$ with Lemma 4.0.1 and a tile structure for this grid structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, c, s)$ with Lemma 4.0.2. Applying Proposition 2.0.2, we obtain a Borel set G' in X with $2\mu(G') \leq \mu(G)$ such that for all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ we have (2.0.22).

LEMMA 4.0.3 (tile sum operator). *We have for all $x \in G \smallsetminus G'$*

$$\sum_{\mathfrak{p} \in \mathfrak{P}} T_{\mathfrak{p}} f(x) = \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x, y) f(y) e(Q(x)(y) - Q(x)(x)) \, d\mu(y). \qquad (4.0.1)$$

PROOF. Fix $x \in G \smallsetminus G'$. Sorting the tiles $\mathfrak{p}$ on the left-hand-side of (4.0.1) by the value $s(\mathfrak{p}) \in [-S, S]$, it suffices to prove for every $-S \leq s \leq S$ that

$$\sum_{\mathfrak{p} \in \mathfrak{P} : s(\mathfrak{p}) = s} T_{\mathfrak{p}} f(x) = 0 \qquad (4.0.2)$$

if $s \notin [\sigma_1(x), \sigma_2(x)]$ and

$$\sum_{\mathfrak{p} \in \mathfrak{P} : s(\mathfrak{p}) = s} T_{\mathfrak{p}} f(x) = \int K_s(x, y) f(y) e(Q(x)(y) - Q(x)(x)) \, d\mu(y). \qquad (4.0.3)$$

if $s \in [\sigma_1(x), \sigma_2(x)]$. If $s \notin [\sigma_1(x), \sigma_2(x)]$, then by definition of $E(\mathfrak{p})$ we have $x \notin E(\mathfrak{p})$ for any $\mathfrak{p}$ with $s(\mathfrak{p}) = s$ and thus $T_{\mathfrak{p}} f(x) = 0$. This proves (4.0.2).

Now assume $s \in [\sigma_1(x), \sigma_2(x)]$. By (2.0.7), (2.0.9), (2.0.10), the fact that $c(I_0) = o$ and $G \subset B(o, \frac{1}{4}D^S)$, there is at least one $I \in \mathcal{D}$ with $s(I) = s$ and $x \in I$. By (2.0.8), this I is unique. By (2.0.13), there is precisely one $\mathfrak{p} \in \mathfrak{P}(I)$ such that $Q(x) \in \Omega(\mathfrak{p})$. Hence there is precisely one $\mathfrak{p} \in \mathfrak{P}$ with $s(\mathfrak{p}) = s$ such that $x \in E(\mathfrak{p})$.

For this $\mathfrak{p}$, the value $T_{\mathfrak{p}}(x)$ by its definition in (2.0.21) equals the right-hand side of (4.0.3). This proves the lemma. $\qquad \square$

We use this to prove Proposition 2.0.1.

PROOF OF PROPOSITION 2.0.1. We now estimate with Lemma 4.0.3 and Proposition 2.0.2

$$\int_{G \smallsetminus G'} \left| \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y) \right| \mathrm{d}\mu(x) \tag{4.0.4}$$

$$= \int_{G \smallsetminus G'} \left| \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y) - Q(x)(x)) \mathrm{d}\mu(y) \right| \mathrm{d}\mu(x) \tag{4.0.5}$$

$$= \int_{G \smallsetminus G'} \left| \sum_{\mathfrak{p} \in \mathfrak{P}} T_{\mathfrak{p}} f(x) \right| \mathrm{d}\mu(x) \leq \frac{2^{440a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}} . \tag{4.0.6}$$

This proves (2.0.6) for the chosen set G' and arbitrary f and thus completes the proof of Proposition 2.0.1. $\qquad \square$

4.1. Proof of Grid Existence Lemma

We begin with the construction of the centers of the dyadic cubes.

LEMMA 4.1.1 (counting balls). *Let $-S \leq k \leq S$. Consider $Y \subset X$ such that for any $y \in Y$, we have*

$$y \in B(o, 4D^S - D^k), \tag{4.1.1}$$

furthermore, for any $y' \in Y$ with $y \neq y'$, we have

$$B(y, D^k) \cap B(y', D^k) = \varnothing. \tag{4.1.2}$$

Then the cardinality of Y is bounded by

$$|Y| \leq 2^{3a+200Sa^3} . \tag{4.1.3}$$

PROOF. Let k and Y be given. By applying the doubling property (1.0.5) inductively, we have for each integer $j \geq 0$

$$\mu(B(y, 2^j D^k)) \leq 2^{aj} \mu(B(y, D^k)) . \tag{4.1.4}$$

Since X is the union of the balls $B(y, 2^j D^k)$ and μ is not zero, at least one of the balls $B(y, 2^j D^k)$ has positive measure, thus $B(y, D^k)$ has positive measure.

Applying (4.1.4) for $j' = \ln_2(8D^{2S}) = 3+2S \cdot 100a^2$ by (2.0.1), using $-S \leq k \leq S$, $y \in B(o, 4D^S)$, and the triangle inequality, we have

$$B(o, 4D^S) \subset B(y, 8D^S) \subset B(y, 2^{j'} D^k) . \tag{4.1.5}$$

Using the disjointedness of the balls in (4.1.2), (4.1.1), and the triangle inequality for ρ, we obtain

$$|Y| \mu(B(o, 4D^S)) \leq 2^{j'a} \sum_{y \in Y} \mu(B(y, D^k)) \tag{4.1.6}$$

$$\leq 2^{j'a} \mu(\bigcup_{y \in Y} B(y, D^k)) \leq 2^{j'a} \mu(o, 4D^S) . \tag{4.1.7}$$

As $\mu(o, 4D^S)$ is not zero, the lemma follows. $\qquad\square$

For each $-S \le k \le S$, let Y_k be a set of maximal cardinality in X such that $Y = Y_k$ satisfies the properties (4.1.1) and (4.1.2) and such that $o \in Y_k$. By the upper bound of Lemma 4.1.1, such a set exists.

For each $-S \le k \le S$, choose an enumeration of the points in the finite set Y_k and thus a total order $<$ on Y_k.

LEMMA 4.1.2 (cover big ball). *For each $-S \le k \le S$, the ball $B(o, 4D^S - D^k)$ is contained in the union of the balls $B(y, 2D^k)$ with $y \in Y_k$.*

PROOF. Let x be any point of $B(o, 4D^S - D^k)$. By maximality of $|Y_k|$, the ball $B(x, D^k)$ intersects one of the balls $B(y, D^k)$ with $y \in Y_k$. By the triangle inequality, $x \in B(y, 2D^k)$. $\qquad\square$

Define the set
$$\mathcal{C} := \{(y, k) : -S \le k \le S, y \in Y_k\} \tag{4.1.8}$$
We totally order the set $\mathcal{C}$ lexicographically by setting $(y, k) < (y', k')$ if $k < k'$ or both $k = k'$ and $y < y'$. In what follows, we define recursively in the sense of this order a function
$$(I_1, I_2, I_3) : \mathcal{C} \to \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X). \tag{4.1.9}$$
Assume the sets $I_j(y', k')$ have already been defined for $j = 1, 2, 3$ if $k' < k$ and if $k = k'$ and $y' < y$.

If $k = -S$, define for $j \in \{1, 2\}$ the set $I_j(y, k)$ to be $B(y, jD^{-S})$. If $-S < k$, define for $j \in \{1, 2\}$ and $y \in Y_k$ the set $I_j(y, k)$ to be

$$\bigcup \{I_3(y', k-1) : y' \in Y_{k-1} \cap B(y, jD^k)\}. \tag{4.1.10}$$

Define for $-S \le k \le S$ and $y \in Y_k$

$$I_3(y, k) := I_1(y, k) \cup \left[I_2(y, k) \smallsetminus \left[X_k \cup \bigcup \{I_3(y', k) : y' \in Y_k, y' < y\}) \right] \right] \tag{4.1.11}$$

with

$$X_k := \bigcup \{I_1(y', k) : y' \in Y_k\}. \tag{4.1.12}$$

LEMMA 4.1.3 (basic grid structure). *For each $-S \le k \le S$ and $1 \le j \le 3$ the following holds.*

If $j \ne 2$ and for some $x \in X$ and $y_1, y_2 \in Y_k$ we have

$$x \in I_j(y_1, k) \cap I_j(y_2, k), \tag{4.1.13}$$

then $y_1 = y_2$.

If $j \ne 1$, then

$$B(o, 4D^S - 2D^k) \subset \bigcup_{y \in Y_k} I_j(y, k). \tag{4.1.14}$$

We have for each $y \in Y_k$,

$$B\left(y, \frac{1}{2}D^k\right) \subset I_3(y, k) \subset B(y, 4D^k). \tag{4.1.15}$$

PROOF. We prove these statements simultaneously by induction on the ordered set of pairs (y, k). Let $-S \leq k \leq S$.

We first consider (4.1.13) for $j = 1$. If $k = -S$, disjointedness of the sets $I_1(y, -S)$ follows by definition of I_1 and Y_k. If $k > -S$, assume x is in $I_1(y_m, k)$ for $m = 1, 2$. Then, for $m = 1, 2$, there is $z_m \in Y_{k-1} \cap B(y_m, D^k)$ with $x \in I_3(z_m, k-1)$. Using (4.1.13) inductively for $j = 3$, we conclude $z_1 = z_2$. This implies that the balls $B(y_1, D^k)$ and $B(y_2, D^k)$ intersect. By construction of Y_k, this implies $y_1 = y_2$. This proves (4.1.13) for $j = 1$.

We next consider (4.1.13) for $j = 3$. Assume x is in $I_3(y_m, k)$ for $m = 1, 2$ and $y_m \in Y_k$. If x is in X_k, then by definition (4.1.11), $x \in I_1(y_m, k)$ for $m = 1, 2$. As we have already shown (4.1.13) for $j = 1$, we conclude $y_1 = y_2$. This completes the proof in case $x \in X_k$, and we may assume x is not in X_k. By definition (4.1.11), x is not in $I_3(z, k)$ for any z with $z < y_1$ or $z < y_2$. Hence, neither $y_1 < y_2$ nor $y_2 < y_1$, and by totality of the order of Y_k, we have $y_1 = y_2$. This completes the proof of (4.1.13) for $j = 3$.

We show (4.1.14) for $j = 2$. In case $k = -S$, this follows from Lemma 4.1.2. Assume $k > -S$. Let x be a point of $B(o, 4D^S - 2D^k)$. By induction, there is $y' \in Y_{k-1}$ such that $x \in I_3(y', k-1)$. Using the inductive statement (4.1.15), we obtain $x \in B(y', 4D^{k-1})$. As $D > 4$, by applying the triangle inequality with the points, o, x, and y', we obtain that $y' \in B(o, 4D^S - D^k)$. By Lemma 4.1.2, y' is in $B(y, 2D^k)$ for some $y \in Y_k$. It follows that $x \in I_2(y, k)$. This proves (4.1.14) for $j = 2$.

We show (4.1.14) for $j = 3$. Let $x \in B(o, 4D^S - 2D^k)$. In case $x \in X_k$, then by definition of X_k we have $x \in I_1(y, k)$ for some $y \in Y_k$ and thus $x \in I_3(y, k)$. We may thus assume $x \notin X_k$. As we have already seen (4.1.14) for $j = 2$, there is $y \in Y_k$ such that $x \in I_2(y, k)$. We may assume this y is minimal with respect to the order in Y_k. Then $x \in I_3(y, k)$. This proves (4.1.14) for $j = 3$.

Next, we show the first inclusion in (4.1.15). Let $x \in B(y, \frac{1}{2}D^k)$. As $I_1(y, k) \subset I_3(y, k)$, it suffices to show $x \in I_1(y, k)$. If $k = -S$, this follows immediately from the assumption on x and the definition of I_1. Assume $k > -S$. By the inductive statement (4.1.14) and $D > 4$, there is a $y' \in Y_{k-1}$ such that $x \in I_3(y', k-1)$. By the inductive statement (4.1.15), we conclude $x \in B(y', 4D^{k-1})$. By the triangle inequality with points x, y, y', and $D \geq 8$, we have $y' \in B(y, D^k)$. It follows by definition (4.1.10) that $I_3(y', k-1) \subset I_1(y, k)$, and thus $x \in I_3(y, k)$. This proves the first inclusion in (4.1.15).

We show the second inclusion in (4.1.15). Let $x \in I_3(y, k)$. As $I_1(y, k) \subset I_2(y, k)$ directly from the definition (4.1.10), it follows by definition (4.1.11) that $x \in I_2(y, k)$. By definition (4.1.10), there is $y' \in Y_{k-1} \cap B(y, 2D^k)$ with $x \in I_3(y', k-1)$. By induction, $x \in B(y', 4D^{k-1})$. By the triangle inequality applied to the points x, y', y and $D > 4$, we conclude $x \in B(y, 4D^k)$. This shows the second inclusion in (4.1.15) and completes the proof of the lemma. $\square$

LEMMA 4.1.4 (cover by cubes). *Let $-S \leq l \leq k \leq S$ and $y \in Y_k$. We have*

$$I_3(y, k) \subset \bigcup_{y' \in Y_l} I_3(y', l) . \tag{4.1.16}$$

PROOF. Let $-S \leq l \leq k \leq S$ and $y \in Y_k$. If $l = k$, the inclusion (4.1.16) is true from the definition of set union. We may then assume inductively that $k > l$ and the statement of the lemma is true if k is replaced by $k - 1$. Let $x \in I_3(y, k)$. By definition (4.1.11), $x \in I_j(y, k)$ for some $j \in \{1, 2\}$. By (4.1.10), $x \in I_3(w, k - 1)$ for some $w \in Y_{k-1}$. We conclude (4.1.16) by induction. $\square$

LEMMA 4.1.5 (dyadic property). *Let $-S \leq l \leq k \leq S$ and $y \in Y_k$ and $y' \in Y_l$ with $I_3(y', l) \cap I_3(y, k) \neq \varnothing$. Then*

$$I_3(y', l) \subset I_3(y, k). \tag{4.1.17}$$

PROOF. Let l, k, y, y' be as in the lemma. Pick $x \in I_3(y', l) \cap I_3(y, k)$. Assume first $l = k$. By (4.1.13) of Lemma 4.1.3, we conclude $y' = y$, and thus (4.1.17). Now assume $l < k$. By induction, we may assume that the statement of the lemma is proven for $k - 1$ in place of k.

By Lemma 4.1.4, there is a $y'' \in Y_{k-1}$ such that $x \in I_3(y'', k - 1)$. By induction, we have $I_3(y', l) \subset I_3(y'', k - 1)$. It remains to prove

$$I_3(y'', k - 1) \subset I_3(y, k). \tag{4.1.18}$$

We make a case distinction and assume first $x \in X_k$. By Definition (4.1.11), we have $x \in I_1(y, k)$. By Definition (4.1.10), there is a $v \in Y_{k-1} \cap B(y, D^k)$ with $x \in I_3(v, k - 1)$. By (4.1.13) of Lemma 4.1.3, we have $v = y''$. By Definition (4.1.10), we then have $I_3(y'', k - 1) \subset I_1(y, k)$. Then (4.1.18) follows by Definition (4.1.11) in the given case.

Assume now the case $x \notin X_k$. By (4.1.11), we have $x \in I_2(y, k)$. Moreover, for any $u < y$ in Y_k, we have $x \notin I_3(u, k)$. Let $u < y$. By transitivity of the order in Y_k, we conclude $x \notin I_2(u, k)$. By (4.1.10) and the disjointedness property of Lemma 4.1.3, we have $I_3(y'', k - 1) \cap I_2(u, k) = \varnothing$. Similarly, $I_3(y'', k - 1) \cap I_1(u, k) = \varnothing$. Hence $I_3(y'', k - 1) \cap I_3(u, k) = \varnothing$. As $u < y$ was arbitrary, we conclude with (4.1.11) the claim in the given case. This completes the proof of (4.1.18), and thus also (4.1.17). $\square$

For $-S \leq k' \leq k \leq S$ and $y' \in Y_{k'}$, $y \in Y_k$ write $(y', k'|y, k)$ if $I_3(y', k') \subset I_3(y, k)$ and

$$\inf_{x \in X \smallsetminus I_3(y, k)} \rho(y', x) < 6 D^{k'}. \tag{4.1.19}$$

LEMMA 4.1.6 (transitive boundary). *Assume $-S \leq k'' < k' < k \leq S$ and $y'' \in Y_{k''}$, $y' \in Y_{k'}$, $y \in Y_k$. Assume there is $x \in X$ such that*

$$x \in I_3(y'', k'') \cap I_3(y', k') \cap I_3(y, k). \tag{4.1.20}$$

If $(y'', k''|y, k)$, the also $(y'', k''|y', k')$ and $(y', k'|y, k)$

PROOF. As $x \in I_3(y'', k'') \cap I_3(y', k')$ and $k'' < k'$, we have by Lemma 4.1.5 that $I_3(y'', k'') \subset I_3(y', k')$. Similarly, $I_3(y', k') \subset I_3(y, k)$. Pick $x' \in X \smallsetminus I_3(y, k)$ such that

$$\rho(y'', x') < 6 D^{k''}, \tag{4.1.21}$$

which exists as $(y'', k''|y, k)$. As $x' \in X \smallsetminus I_3(y', k')$ as well, we conclude $(y'', k''|y', k')$. By the triangle inequality, we have

$$\rho(y', x') \leq \rho(y', x) + \rho(x, y'') + \rho(y'', x') \tag{4.1.22}$$

Using the choice of x and (4.1.15) as well as (4.1.21), we estimate this by

$$< 4D^{k'} + 4D^{k''} + 6D^{k''} \leq 6D^{k'}, \tag{4.1.23}$$

where we have used $D > 5$ and $k'' < k'$. We conclude $(y', k'|y, k)$. $\qquad\square$

LEMMA 4.1.7 (small boundary). *Let* $K = 2^{4a+1}$. *For each* $-S + K \leq k \leq S$ *and* $y \in Y_k$ *we have*

$$\sum_{z \in Y_{k-K}:(z,k-K|y,k)} \mu(I_3(z, k - K)) \leq \frac{1}{2}\mu(I_3(y, k)). \tag{4.1.24}$$

PROOF. Let K be as in the lemma. Let $-S + K \leq k \leq S$ and $y \in Y_k$.

Pick k' so that $k - K \leq k' \leq k$. For each $y'' \in Y_{k-K}$ with $(y'', k - K|y, k)$, by Lemma 4.1.4 and Lemma 4.1.5, there is a unique $y' \in Y_{k'}$ such that

$$I_3(y'', k - K) \subset I_3(y', k') \subset I_3(y, k). \tag{4.1.25}$$

Using Lemma 4.1.6, $(y', k'|y, k)$.

We conclude using the disjointedness property of Lemma 4.1.3 that

$$\sum_{y'':(y'',k-K|y,k)} \mu(I_3(y'', k - K)) \leq \sum_{y':(y',k'|y,k)} \mu(I_3(y', k')). \tag{4.1.26}$$

Adding over $k - K < k' \leq k$, and using

$$\mu(I_3(y', k')) \leq 2^{4a}\mu(B(y', \frac{1}{4}D^{k'}))$$

from the doubling property (1.0.5) and (4.1.15) gives

$$K \sum_{y'':(y'',k-K|y,k)} \mu(I_3(y'', k - K)) \tag{4.1.27}$$

$$\leq 2^{4a} \sum_{k-K<k'\leq k} \left[\sum_{y':(y',k'|y,k)} \mu(B(y', \frac{1}{4}D^{k'})) \right] \tag{4.1.28}$$

Each ball $B(y', \frac{1}{4}D^{k'})$ occurring in (4.1.28) is contained in $I_3(y', k')$ by (4.1.15) and in turn contained in $I_3(y, k)$ by (4.1.25). Assume for the moment all these balls are pairwise disjoint. Then by additivity of the measure,

$$K \sum_{y'':(y'',k-K|y,k)} \mu(I_3(y'', k - K)) \leq 2^{4a}\mu(I_3(y, k)) \tag{4.1.29}$$

which by $K = 2^{4a+1}$ implies (4.1.24).

It thus remains to prove that the balls occurring in (4.1.28) are pairwise disjoint. Let (u, l) and (u', l') be two parameter pairs occurring in the sum of (4.1.28) and let $B(u, \frac{1}{4}D^l)$ and $B(u', \frac{1}{4}D^{l'})$ be the corresponding balls. If $l = l'$, then the balls are equal or disjoint by (4.1.15) and (4.1.13) of Lemma 4.1.3. Assume then without loss of generality that $l' < l$. Towards a contradiction, assume that

$$B(u, \frac{1}{4}D^l) \cap B(u', \frac{1}{4}D^{l'}) \neq \varnothing \tag{4.1.30}$$

As $(u', l'|y, k)$, there is a point x in $X \smallsetminus I_3(y, k)$ with $\rho(x, u') < 6D^{l'}$. Using $D > 25$, we conclude from the triangle inequality and (4.1.30) that $x \in B(u, \frac{1}{2}D^l)$. However,

$B(u, \frac{1}{2}D^l) \subset I_3(u,l)$, and $I_3(u,l) \subset I_3(y,k)$, a contradiction to $x \notin I_3(y,k)$. This proves the lemma. $\square$

LEMMA 4.1.8 (smaller boundary). *Let $K = 2^{4a+1}$ and let $n \geq 0$ be an integer. Then for each $-S + nK \leq k \leq S$ we have*

$$\sum_{y' \in Y_{k-nK}:(y',k-nK|y,k)} \mu(I_3(y', k-nK)) \leq 2^{-n} \mu(I_3(y,k)). \tag{4.1.31}$$

PROOF. We prove this by induction on n. If $n = 0$, both sides of (4.1.31) are equal to $\mu(I_3(y,k))$ by (4.1.13). If $n = 1$, this follows from Lemma 4.1.7.

Assume $n > 1$ and (4.1.31) has been proven for $n - 1$. We write (4.1.31)

$$\sum_{y'' \in Y_{k-nK}:(y'',k-nK|y,k)} \mu(I_3(y'', k-nK)) \tag{4.1.32}$$

$$= \sum_{y' \in Y_{k-K}:(y',k-K|y,k)} \left[\sum_{y'' \in Y_{k-nK}:(y'',k-nK|y',k-K)} \mu(I_3(y'', k-nK)) \right] \tag{4.1.33}$$

Applying the induction hypothesis, this is bounded by

$$= \sum_{y' \in Y_{k-K}:(y',k-K|y,k)} 2^{1-n} \mu(I_3(y', k-K)) \tag{4.1.34}$$

Applying (4.1.24) gives (4.1.31), and proves the lemma. $\square$

LEMMA 4.1.9 (boundary measure). *For each $-S \leq k \leq S$ and $y \in Y_k$ and $0 < t < 1$ with $tD^k \geq D^{-S}$ we have*

$$\mu(\{x \in I_3(y,k) \ : \ \rho(x, X \smallsetminus I_3(y,k)) \leq tD^k\}) \leq 2t^\kappa \mu(I_3(y,k)). \tag{4.1.35}$$

PROOF. Let $x \in I_3(y,k)$ with $\rho(x, X \smallsetminus I_3(y,k)) \leq tD^k$. Let $K = 2^{4a+1}$ as in Lemma 4.1.8. Let n be the largest integer such that $D^{nK} \leq \frac{1}{t}$, so that $tD^k \leq D^{k-nK}$ and

$$D^{nK} > \frac{1}{tD^K}. \tag{4.1.36}$$

Let $k' = k - nK$, by the assumption $tD^k \geq D^{-S}$, we have $k' \geq -S$. By (4.1.16), there exists $y' \in Y_{k'}$ with $x \in I_3(y', k')$. By the squeezing property (4.1.15) and the assumption on x, we have

$$\rho(y', X \smallsetminus I_3(y,k)) \leq \rho(x,y') + \rho(x, X \smallsetminus I_3(y,k)) \leq 4D^{k'} + tD^k.$$

By the assumption on n and the definition of k', this is

$$\leq 4D^{k'} + D^{k-nK} < 6D^{k'}.$$

Together with (4.1.17) thus $(y', k'|y, k)$. We have shown that

$$\{x \in I_3(y,k) \ : \ \rho(x, X \smallsetminus I_3(y,k)) \leq tD^k\}$$

$$\subset \bigcup_{y' \in Y_{k-nK}:(y',k-nK|y,k)} I_3(y', k-nK).$$

Using monotonicity and additivity of the measure and Lemma 4.1.8, we obtain

$$\mu(\{x \in I_3(y,k) \ : \ \rho(x, X \smallsetminus I_3(y,k)) \leq tD^k\}) \leq 2^{-n} \mu(I_3(y,k)).$$

By (4.1.36) and the definition (2.0.1) of D, this is bounded by

$$2t^{1/(100a^2 K)} \mu(I_3(y,k)),$$

which completes the proof by the definition (2.0.2) of κ. $\square$

Let $\tilde{\mathcal{D}}$ be the set of all $I_3(y,k)$ with $k \in [-S, S]$ and $y \in Y_k$. Define

$$s(I_3(y,k)) := k \tag{4.1.37}$$

$$c(I_3(y,k)) := y. \tag{4.1.38}$$

We define $\mathcal{D}$ to be the set of all $I \in \tilde{\mathcal{D}}$ such that $I \subset I_3(o, S)$.

PROOF OF LEMMA 4.0.1. We first show that $(\tilde{\mathcal{D}}, c, s)$ satisfies properties (2.0.7), (2.0.8), (2.0.10) and (2.0.11). Property (2.0.10) follows from (4.1.15), while (2.0.11) follows from Lemma 4.1.9.

Let $x \in B(o, D^S)$. We show properties (2.0.7) and (2.0.8) for $(\tilde{\mathcal{D}}, c, s)$ and x.

We first show (2.0.7) for $(\tilde{\mathcal{D}}, c, s)$ by contradiction. Then there is an I violating the conclusion of (2.0.7). Pick such $I = I_3(y,l)$ such that l is minimal. By assumption, we have $-S \le k < l$; in particular $-S < l$. By definition, $I_3(y,l)$ is contained in $I_1(y,l) \cup I_2(y,l)$, which is contained in the union of $I_3(y', l-1)$ with $y' \in Y_{l-1}$. By minimality of l, each such $I_3(y', l-1)$ is contained in the union of all $I_3(z,k)$ with $z \in Y_k$. This proves (2.0.7).

We now show (2.0.8) for $(\tilde{\mathcal{D}}, c, s)$. Assume to get a contradiction that there are non-disjoint $I, J \in \tilde{\mathcal{D}}$ with $s(I) \le s(J)$ and $I \not\subset J$. We may assume the existence of such I and J with minimal $s(J) - s(I)$. Let $k = s(I)$. Assume first $s(J) = k$. Let $I = I_3(y_1, k)$ and $J = I_3(y_2, k)$ with $y_1, y_2 \in Y_k$. If $y_1 = y_2$, then $I = J$, a contradiction to $I \not\subset J$. If $y_1 \ne y_2$, then $I \cap J = \varnothing$ by (4.1.13), a contradiction to the non-disjointedness of I, J. Assume now $s(J) > k$. Choose $y \in I \cap J$. By property (2.0.7), there is $K \in \tilde{\mathcal{D}}$ with $s(K) = s(J) - 1$ and $y \in K$. By construction of J, and pairwise disjointedness of all $I_3(w, s(J)-1)$ that we have already seen, we have $K \subset J$. By minimality of $s(J)$, we have $I \subset K$. This proves $I \subset J$ and thus (2.0.8).

Now note that properties (2.0.8), (2.0.10) and (2.0.11) immediately carry over to $(\mathcal{D}, c, s)$ by restriction. (2.0.9) is true for $(\mathcal{D}, c, s)$ by definition, and (2.0.7) follows from (2.0.7) and (2.0.8) for $(\tilde{\mathcal{D}}, c, s)$. $\square$

4.2. Proof of Tile Structure Lemma

Choose a grid structure $(\mathcal{D}, c, s)$ with Lemma 4.0.1 Let $I \in \mathcal{D}$. Suppose that

$$\mathcal{Z} \subset Q(X) \tag{4.2.1}$$

is such that for any $\vartheta, \theta \in \mathcal{Z}$ with $\vartheta \ne \theta$ we have

$$B_{I^\circ}(\vartheta, 0.3) \cap B_{I^\circ}(\theta, 0.3) \cap Q(X) = \varnothing. \tag{4.2.2}$$

Since $Q(X)$ is finite, there exists a set $\mathcal{Z}$ satisfying both (4.2.1) and (4.2.2) of maximal cardinality among all such sets. We pick for each $I \in \mathcal{D}$ such a set $\mathcal{Z}(I)$.

LEMMA 4.2.1 (frequency ball cover). *For each $I \in \mathcal{D}$, we have*

$$Q(X) \subset \bigcup_{z \in \mathcal{Z}(I)} B_{I^\circ}(z, 0.7). \tag{4.2.3}$$

PROOF. Let $\theta \in \bigcup_{\vartheta \in Q(X)} B_{I^\circ}(\vartheta, 1)$. By maximality of $\mathcal{Z}(I)$, there must be a point $z \in \mathcal{Z}(I)$ such that $B_{I^\circ}(z, 0.3) \cap B_{I^\circ}(\theta, 0.3) \neq \varnothing$. Else, $\mathcal{Z}(I) \cup \{\theta\}$ would be a set of larger cardinality than $\mathcal{Z}(I)$ satisfying (4.2.1) and (4.2.2). Fix such z, and fix a point $z_1 \in B_{I^\circ}(z, 0.3) \cap B_{I^\circ}(\theta, 0.3)$. By the triangle inequality, we deduce that

$$d_{I^\circ}(z, \theta) \leq d_{I^\circ}(z, z_1) + d_{I^\circ}(\theta, z_1) < 0.3 + 0.3 = 0.6,$$

and hence $\theta \in B_{I^\circ}(z, 0.7)$. $\square$

We define

$$\mathfrak{P} = \{(I, z) \ : \ I \in \mathcal{D}, z \in \mathcal{Z}(I)\},$$

$$\mathcal{I}((I, z)) = I \qquad \text{and} \qquad \mathcal{Q}((I, z)) = z.$$

We further set

$$\mathrm{s}(\mathfrak{p}) = s(\mathcal{I}(\mathfrak{p})), \qquad \mathrm{c}(\mathfrak{p}) = c(\mathcal{I}(\mathfrak{p})).$$

Then (2.0.18), (2.0.19) hold by definition.

It remains to construct the map Ω, and verify properties (2.0.13), (2.0.14) and (2.0.15). We first construct an auxiliary map Ω_1. For each $I \in \mathcal{D}$, we pick an enumeration of the finite set $\mathcal{Z}(I)$

$$\mathcal{Z}(I) = \{z_1, \dots, z_M\}.$$

We define $\Omega_1 : \mathfrak{P} \mapsto \mathcal{P}(\Theta)$ as below. Set

$$\Omega_1((I, z_1)) = B_{I^\circ}(z_1, 0.7) \smallsetminus \bigcup_{z \in \mathcal{Z}(I) \smallsetminus \{z_1\}} B_{I^\circ}(z, 0.3)$$

and then define iteratively

$$\Omega_1((I, z_k)) = B_{I^\circ}(z_k, 0.7) \smallsetminus \bigcup_{z \in \mathcal{Z}(I) \smallsetminus \{z_k\}} B_{I^\circ}(z, 0.3) \smallsetminus \bigcup_{i=1}^{k-1} \Omega_1((I, z_i)). \tag{4.2.4}$$

LEMMA 4.2.2 (disjoint frequency cubes). *For each $I \in \mathcal{D}$, and $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}(I)$, if*

$$\Omega_1(\mathfrak{p}_1) \cap \Omega_1(\mathfrak{p}_2) \neq \varnothing,$$

then $\mathfrak{p}_1 = \mathfrak{p}_2$.

PROOF. By the definition of the map $\mathcal{I}$, we have

$$\mathfrak{P}(I) = \{(I, z) \ : \ z \in \mathcal{Z}(I)\}.$$

By (4.2.4), the set $\Omega_1((I, z_k))$ is disjoint from each $\Omega_1((I, z_i))$ with $i < k$. Thus the sets $\Omega_1(\mathfrak{p})$, $\mathfrak{p} \in \mathfrak{P}(I)$ are pairwise disjoint. $\square$

LEMMA 4.2.3 (frequency cube cover). *For each $I \in \mathcal{D}$, it holds that*

$$\bigcup_{z \in \mathcal{Z}(I)} B_{I^\circ}(z, 0.7) \subset \bigcup_{\mathfrak{p} \in \mathfrak{P}(I)} \Omega_1(\mathfrak{p}) \,. \tag{4.2.5}$$

For every $\mathfrak{p} \in \mathfrak{P}$, it holds that

$$B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.3) \subset \Omega_1(\mathfrak{p}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.7) \,. \tag{4.2.6}$$

PROOF. For (4.2.6) let $\mathfrak{p} = (I, z)$. The second inclusion in (4.2.6) then follows from (4.2.4) and the equality $B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.7) = B_{I^\circ}(z, 0.7)$, which is true by definition. For the first inclusion in (4.2.6) let $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.3)$. Let k be such that $z = z_k$ in the enumeration we chose above. It follows immediately from (4.2.4) and (4.2.2) that $\vartheta \notin \Omega_1((I, z_i))$ for all $i < k$. Thus, again from (4.2.4), we have $\vartheta \in \Omega_1((I, z_k))$.

To show (4.2.5) let $\vartheta \in \bigcup_{z \in \mathcal{Z}(I)} B_{I^\circ}(z, 0.7)$. If there exists $z \in \mathcal{Z}(I)$ with $\vartheta \in B_{I^\circ}(z, 0.3)$, then

$$z \in \Omega_1((I, z)) \subset \bigcup_{\mathfrak{p} \in \mathfrak{P}(I)} \Omega_1(\mathfrak{p})$$

by the first inclusion in (4.2.6).

Now suppose that there exists no $z \in \mathcal{Z}(I)$ with $\vartheta \in B_{I^\circ}(z, 0.3)$. Let k be minimal such that $\vartheta \in B_{I^\circ}(z_k, 0.7)$. Since $\Omega_1((I, z_i)) \subset B_{I^\circ}(z_i, 0.7)$ for each i by (4.2.4), we have that $\vartheta \notin \Omega_1((I, z_i))$ for all $i < k$. Hence $\vartheta \in \Omega_1((I, z_k))$, again by (4.2.4). □

Now we are ready to define the function Ω. We define for all $\mathfrak{p} \in \mathfrak{P}(I_0)$

$$\Omega(\mathfrak{p}) = \Omega_1(\mathfrak{p}) \,. \tag{4.2.7}$$

For all other cubes $I \in \mathcal{D}$, $I \neq I_0$, there exists, by (2.0.8) and (2.0.9), $J \in \mathcal{D}$ with $I \subset J$ and $I \neq J$. On such I we define Ω by recursion. We can pick an inclusion minimal $J \in \mathcal{D}$ among the finitely many cubes such that $I \subset J$ and $I \neq J$. This J is unique: Suppose that J' is another inclusion minimal cube with $I \subset J'$ and $I \neq J'$. Without loss of generality, we have that $s(J) \leq s(J')$. By (2.0.8), it follows that $J \subset J'$. Since J' is minimal with respect to inclusion, it follows that $J = J'$. Then we define

$$\Omega(\mathfrak{p}) = \bigcup_{z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p})} \Omega((J, z)) \cup B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \,. \tag{4.2.8}$$

We now verify that $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$ forms a tile structure.

PROOF OF LEMMA 4.0.2. First, we prove (2.0.15). If $I = I_0$, then (2.0.15) holds for all $\mathfrak{p} \in \mathfrak{P}(I)$ by (4.2.7) and (4.2.6). Now suppose that I is not maximal in $\mathcal{D}$ with respect to set inclusion. Then we may assume by induction that for all $J \in \mathcal{D}$ with $I \subset J$ and all $\mathfrak{p}' \in \mathfrak{P}(J)$, (2.0.15) holds. Let J be the unique minimal cube in $\mathcal{D}$ with $I \subsetneq J$.

Suppose that $\vartheta \in \Omega(\mathfrak{p})$. If $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2)$, then since

$$B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 1) \,,$$

we conclude that $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.7)$. If not, by (4.2.8), there exists $z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p})$ with $\vartheta \in \Omega(J, z)$. Using the triangle inequality and (4.2.6), we obtain

$$d_{I^\circ}(\mathcal{Q}(\mathfrak{p}), \vartheta) \leq d_{I^\circ}(\mathcal{Q}(\mathfrak{p}), z) + d_{I^\circ}(z, \vartheta) \leq 0.7 + d_{I^\circ}(z, \vartheta) \,.$$

By Lemma 2.1.2 and the induction hypothesis, this is estimated by

$$\leq 0.7 + 2^{-95a} d_{J^\circ}(z, \vartheta) \leq 0.7 + 2^{-95a} \cdot 1 < 1 \, .$$

This shows the second inclusion in (2.0.15). The first inclusion is immediate from (4.2.8).

Next, we show (2.0.13). Let $I \in \mathcal{D}$.

If $I = I_0$, then disjointedness of the sets $\Omega(\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{P}(I)$ follows from the definition (4.2.7) and Lemma 4.2.2. To obtain the inclusion in (2.0.13) one combines the inclusions (4.2.3) and (4.2.5) of Lemma 4.2.3 with (4.2.7).

Now we turn to the case where there exists $J \in \mathcal{D}$ with $I \subset J$ and $I \neq J$. In this case we use induction: It suffices to show (2.0.13) under the assumption that it holds for all cubes $J \in \mathcal{D}$ with $I \subset J$. As shown before definition (4.2.8), we may choose the unique inclusion minimal such J. To show disjointedness of the sets $\Omega(\mathfrak{p}), \mathfrak{p} \in \mathfrak{P}(I)$ we pick two tiles $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}(I)$ and $\vartheta \in \Omega(\mathfrak{p}) \cap \Omega(\mathfrak{p}')$. Then we are by (4.2.8) in one of the following four cases.

1. There exist $z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p})$ such that $\vartheta \in \Omega(J, z)$, and there exists $z' \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p}')$ such that $\vartheta \in \Omega(J, z')$. By the induction hypothesis, that (2.0.13) holds for J, we must have $z = z'$. By Lemma 4.2.2, we must then have $\mathfrak{p} = \mathfrak{p}'$.

2. There exists $z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p})$ such that $\vartheta \in \Omega(J, z)$, and $\vartheta \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 0.2)$. Using the triangle inequality, Lemma 2.1.2 and (2.0.15), we obtain

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), z) \leq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta) + d_{\mathfrak{p}'}(z, \vartheta) \leq 0.2 + 2^{-95a} \cdot 1 < 0.3 \, .$$

Thus $z \in \Omega_1(\mathfrak{p}')$ by (4.2.6). By Lemma 4.2.2, it follows that $\mathfrak{p} = \mathfrak{p}'$.

3. There exists $z' \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p}')$ such that $\vartheta \in \Omega(J, z')$, and $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2)$. This case is the same as case 2., after swapping $\mathfrak{p}$ and $\mathfrak{p}'$.

4. We have $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \cap B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 0.2)$. In this case it follows that $\mathfrak{p} = \mathfrak{p}'$ since the sets $B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2)$ are pairwise disjoint by the inclusion (4.2.6) and Lemma 4.2.2.

To show the inclusion in (2.0.13), let $\vartheta \in Q(X)$. By the induction hypothesis, there exists $\mathfrak{p} \in \mathfrak{P}(J)$ such that $\vartheta \in \Omega(\mathfrak{p})$. By definition of the set $\mathfrak{P}$, we have $\mathfrak{p} = (J, z)$ for some $z \in \mathcal{Z}(J)$. Thus, by (4.2.3), there exists $z' \in \mathcal{Z}(I)$ with $z \in B_{I^\circ}(z', 0.7)$. Then by Lemma (4.2.3) there exists $\mathfrak{p}' \in \mathfrak{P}(I)$ with $z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p}')$. Consequently, by (4.2.8), $\vartheta \in \Omega(\mathfrak{p}')$. This completes the proof of (2.0.13).

Finally, we show (2.0.14). Let $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{q})$ and $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) \neq \varnothing$. If we have $s(\mathfrak{p}) \geq s(\mathfrak{q})$, then it follows from (2.0.8) that $I = J$, thus $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}(I)$. By (2.0.13) we have then either $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) = \varnothing$ or $\Omega(\mathfrak{p}) = \Omega(\mathfrak{q})$. By the assumption in (2.0.14) we have $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) \neq \varnothing$, so we must have $\Omega(\mathfrak{p}) = \Omega(\mathfrak{q})$ and in particular $\Omega(\mathfrak{q}) \subset \Omega(\mathfrak{p})$.

So it remains to show (2.0.14) under the additional assumption that $s(\mathfrak{q}) > s(\mathfrak{p})$. In this case, we argue by induction on $s(\mathfrak{q}) - s(\mathfrak{p})$. By (2.0.7), there exists a cube $J \in \mathcal{D}$ with $s(J) = s(\mathfrak{q}) - 1$ and $J \cap \mathcal{I}(\mathfrak{p}) \neq \varnothing$. We pick one such J. By (2.0.8), we have $\mathcal{I}(\mathfrak{p}) \subset J \subset \mathcal{I}(\mathfrak{q})$.

Thus, by (4.2.3), there exists $z' \in \mathcal{Z}(J)$ with $\mathcal{Q}(\mathfrak{q}) \in B_{J^\circ}(z', 0.7)$. Then by Lemma 4.2.3 there exists $\mathfrak{q}' \in \mathfrak{P}(J)$ with $\mathcal{Q}(\mathfrak{q}) \in \Omega_1(\mathfrak{q}')$. By (4.2.8), it follows that $\Omega(\mathfrak{q}) \subset \Omega(\mathfrak{q}')$. Note that then $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{q}')$ and $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}') \neq \varnothing$ and $s(\mathfrak{q}') - s(\mathfrak{p}) =$

$s(\mathfrak{q}) - s(\mathfrak{p}) - 1$. Thus, we have by the induction hypothesis that $\Omega(\mathfrak{q}') \subset \Omega(\mathfrak{p})$. This completes the proof. □

Proof of discrete Carleson

Let a grid structure $(\mathcal{D}, c, s)$ and a tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q})$ for this grid structure be given. In Section 5.1, we decompose the set $\mathfrak{P}$ of tiles into subsets. Each subset will be controlled by one of three methods. The guiding principle of the decomposition is to be able to apply the forest estimate of Proposition 2.0.4 to the final subsets defined in (5.1.23). This application is done in Section 5.4. The miscellaneous subsets along the construction of the forests will either be thrown into exceptional sets, which are defined and controlled in Section 5.2, or will be controlled by the antichain estimate of Proposition 2.0.3, which is done in Section 5.5. Section 5.3 contains some auxiliary lemmas needed for the proofs in Subsections 5.4-5.5.

5.1. Organisation of the tiles

In the following definitions, k, n, and j will be nonnegative integers. Define $\mathcal{C}(G, k)$ to be the set of $I \in \mathcal{D}$ such that there exists a $J \in \mathcal{D}$ with $I \subset J$ and

$$\mu(G \cap J) > 2^{-k-1}\mu(J), \tag{5.1.1}$$

but there does not exist a $J \in \mathcal{D}$ with $I \subset J$ and

$$\mu(G \cap J) > 2^{-k}\mu(J). \tag{5.1.2}$$

Let

$$\mathfrak{P}(k) = \{\mathfrak{p} \in \mathfrak{P} \ : \ \mathcal{I}(\mathfrak{p}) \in \mathcal{C}(G, k)\} \tag{5.1.3}$$

Define $\mathfrak{M}(k, n)$ to be the set of $\mathfrak{p} \in \mathfrak{P}(k)$ such that

$$\mu(E_1(\mathfrak{p})) > 2^{-n}\mu(\mathcal{I}(\mathfrak{p})) \tag{5.1.4}$$

and there does not exist $\mathfrak{p}' \in \mathfrak{P}(k)$ with $\mathfrak{p}' \neq \mathfrak{p}$ and $\mathfrak{p} \leq \mathfrak{p}'$ such that

$$\mu(E_1(\mathfrak{p}')) > 2^{-n}\mu(\mathcal{I}(\mathfrak{p}')). \tag{5.1.5}$$

Define for a collection $\mathfrak{P}' \subset \mathfrak{P}(k)$

$$\mathrm{dens}'_k(\mathfrak{P}') := \sup_{\mathfrak{p}' \in \mathfrak{P}'} \sup_{\lambda \geq 2} \lambda^{-a} \sup_{\mathfrak{p} \in \mathfrak{P}(k): \lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}} \frac{\mu(E_2(\lambda, \mathfrak{p}))}{\mu(\mathcal{I}(\mathfrak{p}))} . \tag{5.1.6}$$

Sorting by density, we define

$$\mathfrak{C}(k, n) := \{\mathfrak{p} \in \mathfrak{P}(k) \ : \ 2^{4a}2^{-n} < \mathrm{dens}'_k(\{\mathfrak{p}\}) \leq 2^{4a}2^{-n+1}\} . \tag{5.1.7}$$

Following Fefferman [Fef73], we define for $\mathfrak{p} \in \mathfrak{C}(k, n)$

$$\mathfrak{B}(\mathfrak{p}) := \{\mathfrak{m} \in \mathfrak{M}(k, n) \ : \ 100\mathfrak{p} \lesssim \mathfrak{m}\} \tag{5.1.8}$$

and
$$\mathfrak{C}_1(k,n,j) := \left\{ \mathfrak{p} \in \mathfrak{C}(k,n) \ : \ 2^j \leq |\mathfrak{B}(\mathfrak{p})| < 2^{j+1} \right\}. \tag{5.1.9}$$

and
$$\mathfrak{L}_0(k,n) := \left\{ \mathfrak{p} \in \mathfrak{C}(k,n) \ : \ |\mathfrak{B}(\mathfrak{p})| < 1 \right\}. \tag{5.1.10}$$

Together with the following removal of minimal layers, the splitting into $\mathfrak{C}_1(k,n,j)$ will lead to a separation of trees. Define recursively for $0 \leq l \leq Z(n+1)$

$$\mathfrak{L}_1(k,n,j,l) \tag{5.1.11}$$

to be the set of minimal elements with respect to $\leq$ in

$$\mathfrak{C}_1(k,n,j) \smallsetminus \bigcup_{0 \leq l' < l} \mathfrak{L}_1(k,n,j,l'). \tag{5.1.12}$$

Define
$$\mathfrak{C}_2(k,n,j) := \mathfrak{C}_1(k,n,j) \smallsetminus \bigcup_{0 \leq l' \leq Z(n+1)} \mathfrak{L}_1(k,n,j,l'). \tag{5.1.13}$$

The remaining tile organization will be relative to prospective tree tops, which we define now. Define

$$\mathfrak{U}_1(k,n,j) \tag{5.1.14}$$

to be the set of all $\mathfrak{u} \in \mathfrak{C}_1(k,n,j)$ such that for all $\mathfrak{p} \in \mathfrak{C}_1(k,n,j)$ with $\mathcal{I}(\mathfrak{u})$ strictly contained in $\mathcal{I}(\mathfrak{p})$ we have $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}),100) \cap B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}),100) = \varnothing$.

We first remove the pairs that are outside the immediate reach of any of the prospective tree tops. Define

$$\mathfrak{L}_2(k,n,j) \tag{5.1.15}$$

to be the set of all $\mathfrak{p} \in \mathfrak{C}_2(k,n,j)$ such that there does not exist $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u})$ and $2\mathfrak{p} \lesssim \mathfrak{u}$. Define

$$\mathfrak{C}_3(k,n,j) := \mathfrak{C}_2(k,n,j) \smallsetminus \mathfrak{L}_2(k,n,j). \tag{5.1.16}$$

We next remove the maximal layers. Define recursively for $0 \leq l \leq Z(n+1)$

$$\mathfrak{L}_3(k,n,j,l) \tag{5.1.17}$$

to be the set of all maximal elements with respect to $\leq$ in

$$\mathfrak{C}_3(k,n,j) \smallsetminus \bigcup_{0 \leq l' < l} \mathfrak{L}_3(k,n,j,l'). \tag{5.1.18}$$

Define
$$\mathfrak{C}_4(k,n,j) := \mathfrak{C}_3(k,n,j) \smallsetminus \bigcup_{0 \leq l \leq Z(n+1)} \mathfrak{L}_3(k,n,j,l). \tag{5.1.19}$$

Finally, we remove the boundary pairs relative to the prospective tree tops. Define

$$\mathcal{L}(\mathfrak{u}) \tag{5.1.20}$$

to be the set of all $I \in \mathcal{D}$ with $I \subset \mathcal{I}(\mathfrak{u})$ and $s(I) = \mathrm{s}(\mathfrak{u}) - Z(n+1) - 1$ and

$$B(c(I), 8D^{s(I)}) \not\subset \mathcal{I}(\mathfrak{u}). \tag{5.1.21}$$

Define

$$\mathfrak{L}_4(k,n,j) \tag{5.1.22}$$

to be the set of all $\mathfrak{p} \in \mathfrak{C}_4(k, n, j)$ such that there exists $\mathfrak{u} \in \mathfrak{U}_1(k, n, j)$ with $\mathcal{J}(\mathfrak{p}) \subset \bigcup \mathcal{L}(\mathfrak{u})$, and define

$$\mathfrak{C}_5(k, n, j) := \mathfrak{C}_4(k, n, j) \smallsetminus \mathfrak{L}_4(k, n, j) \,. \tag{5.1.23}$$

We define three exceptional sets. The first exceptional set G_1 takes into account the ratio of the measures of F and G. Define $\mathfrak{P}_{F,G}$ to be the set of all $\mathfrak{p} \in \mathfrak{P}$ with

$$\operatorname{dens}_2(\{\mathfrak{p}\}) > 2^{2a+5} \frac{\mu(F)}{\mu(G)} \,. \tag{5.1.24}$$

Define

$$G_1 := \bigcup_{\mathfrak{p} \in \mathfrak{P}_{F,G}} \mathcal{J}(\mathfrak{p}) \,. \tag{5.1.25}$$

For an integer $\lambda \geq 0$, define $A(\lambda, k, n)$ to be the set of all $x \in X$ such that

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n)} \mathbf{1}_{\mathcal{J}(\mathfrak{p})}(x) > \lambda 2^{n+1} \tag{5.1.26}$$

and define

$$G_2 := \bigcup_{k \geq 0} \bigcup_{k < n} A(2n + 6, k, n) \,. \tag{5.1.27}$$

Define

$$G_3 := \bigcup_{k \geq 0} \bigcup_{n \geq k} \bigcup_{0 \leq j \leq 2n+3} \bigcup_{\mathfrak{p} \in \mathfrak{L}_4(k,n,j)} \mathcal{J}(\mathfrak{p}) \,. \tag{5.1.28}$$

Define $G' = G_1 \cup G_2 \cup G_3$ The following bound of the measure of G' will be proven in Section 5.2.

LEMMA 5.1.1 (exceptional set). *We have*

$$\mu(G') \leq 2^{-2} \mu(G) \,. \tag{5.1.29}$$

In Section 5.4, we identify each set $\mathfrak{C}_5(k, n, j)$ outside G' as forest and use Proposition 2.0.4 to prove the following lemma.

LEMMA 5.1.2 (forest union). *Let*

$$\mathfrak{P}_1 = \bigcup_{k \geq 0} \bigcup_{n \geq k} \bigcup_{0 \leq j \leq 2n+3} \mathfrak{C}_5(k, n, j) \tag{5.1.30}$$

For all $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ we have

$$\int_{G \smallsetminus G'} \left| \sum_{\mathfrak{p} \in \mathfrak{P}_1} T_{\mathfrak{p}} f \right| \mathrm{d}\mu \leq \frac{2^{435a^3}}{(q-1)^4} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}} \,. \tag{5.1.31}$$

In Section 5.5, we decompose the complement of the set of tiles in Lemma 5.1.2 and apply the antichain estimate of Proposition 2.0.3 to prove the following lemma.

LEMMA 5.1.3 (forest complement). *Let*

$$\mathfrak{P}_2 = \mathfrak{P} \smallsetminus \mathfrak{P}_1 \,. \tag{5.1.32}$$

For all $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ we have

$$\int_{G \smallsetminus G'} \left| \sum_{\mathfrak{p} \in \mathfrak{P}_2} T_{\mathfrak{p}} f \right| \mathrm{d}\mu \leq \frac{2^{210a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}} \,. \tag{5.1.33}$$

PROOF OF PROPOSITION 2.0.2. Proposition 2.0.2 follows by applying the triangle inequality to (2.0.22) according to the splitting in Lemma 5.1.2 and Lemma 5.1.3 and using both Lemmas as well as the bound on the set G' given by Lemma 5.1.1.   □

5.2. Proof of the Exceptional Sets Lemma

We prove separate bounds for G_1, G_2, and G_3 in Lemmas 5.2.1, 5.2.6, and 5.2.10. Adding up these bounds proves Lemma 5.1.1.

The bound for G_1 follows from the Vitali covering lemma, Proposition 2.0.6.

LEMMA 5.2.1 (first exception). *We have*

$$\mu(G_1) \leq 2^{-5}\mu(G)\,. \tag{5.2.1}$$

PROOF. Let

$$K = 2^{2a+5}\frac{\mu(F)}{\mu(G)}\,.$$

For each $\mathfrak{p} \in \mathfrak{P}_{F,G}$ pick a $r(\mathfrak{p}) > 4D^{\mathrm{s}(\mathfrak{p})}$ with

$$\mu(F \cap B(\mathrm{c}(\mathfrak{p}), r(\mathfrak{p}))) \geq K\mu(B(\mathrm{c}(\mathfrak{p}), r(\mathfrak{p})))\,.$$

This ball exists by definition of $\mathfrak{P}_{F,G}$ and dens_2. By applying Proposition 2.0.6 to the collection of balls

$$\mathcal{B} = \{B(\mathrm{c}(\mathfrak{p}), r(\mathfrak{p})) \ : \ \mathfrak{p} \in \mathfrak{P}_{F,G}\}$$

and the function $u = \mathbf{1}_F$, we obtain

$$\mu(\bigcup \mathcal{B}) \leq 2^{2a}K^{-1}\mu(F)\,.$$

We conclude with (2.0.10) and $r(\mathfrak{p}) > 4D^{\mathrm{s}(\mathfrak{p})}$

$$\mu(G_1) = \mu(\bigcup_{\mathfrak{p} \in \mathfrak{P}_{F,G}} \mathcal{J}(\mathfrak{p})) \leq \mu(\bigcup \mathcal{B}) \leq 2^{2a}K^{-1}\mu(F) = 2^{-5}\mu(G)\,.$$

□

We turn to the bound of G_2, which relies on the Dyadic Covering Lemma 5.2.2 and the John-Nirenberg Lemma 5.2.5 below.

LEMMA 5.2.2 (dense cover). *For each $k \geq 0$, the union of all dyadic cubes in $\mathcal{C}(G, k)$ has measure at most $2^{k+1}\mu(G)$.*

PROOF. The union of dyadic cubes in $\mathcal{C}(G, k)$ is contained the union of elements of the set $\mathcal{M}(k)$ of all dyadic cubes J with $\mu(G \cap J) > 2^{-k-1}\mu(J)$. The union of elements in the set $\mathcal{M}(k)$ is contained in the union of elements in the set $\mathcal{M}^*(k)$ of maximal elements in $\mathcal{M}(k)$ with respect to set inclusion. Hence

$$\mu(\bigcup \mathcal{C}(G, k)) \leq \mu(\bigcup \mathcal{M}^*(k)) \leq \sum_{J \in \mathcal{M}^*(k)} \mu(J) \tag{5.2.2}$$

Using the definition of $\mathcal{M}(k)$ and then the pairwise disjointedness of elements in $\mathcal{M}^*(k)$, we estimate (5.2.2) by

$$\leq 2^{k+1} \sum_{J \in \mathcal{M}^*(k)} \mu(J \cap G) \leq 2^{k+1}\mu(G). \tag{5.2.3}$$

This proves the lemma. $\qquad\square$

LEMMA 5.2.3 (pairwise disjoint). *If* $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{M}(k,n)$ *and*

$$E_1(\mathfrak{p}) \cap E_1(\mathfrak{p}') \neq \varnothing, \qquad (5.2.4)$$

then $\mathfrak{p} = \mathfrak{p}'$.

PROOF. Let $\mathfrak{p}, \mathfrak{p}'$ be as in the lemma. By definition of E_1, we have $E_1(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{p})$ and analogously for $\mathfrak{p}'$, we conclude from (5.2.4) that $\mathcal{J}(\mathfrak{p}) \cap \mathcal{J}(\mathfrak{p}') \neq \varnothing$. Let without loss of generality $\mathcal{J}(\mathfrak{p})$ be maximal in $\{\mathcal{J}(\mathfrak{p}), \mathcal{J}(\mathfrak{p}')\}$, then $\mathcal{J}(\mathfrak{p}') \subset \mathcal{J}(\mathfrak{p})$. By (5.2.4), we conclude by definition of E_1 that $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{p}') \neq \varnothing$. By (2.0.14) we conclude $\Omega(\mathfrak{p}) \subset \Omega(\mathfrak{p}')$. It follows that $\mathfrak{p}' \leq \mathfrak{p}$. By maximality (5.1.5) of $\mathfrak{p}'$, we have $\mathfrak{p}' = \mathfrak{p}$. This proves the lemma. $\qquad\square$

LEMMA 5.2.4 (dyadic union). *For each* $x \in A(\lambda, k, n)$, *there is a dyadic cube* I *that contains* x *and is a subset of* $A(\lambda, k, n)$.

PROOF. Fix k, n, λ, x as in the lemma such that $x \in A(\lambda, k, n)$. Let $\mathcal{M}$ be the set of dyadic cubes $\mathcal{J}(\mathfrak{p})$ with $\mathfrak{p}$ in $\mathfrak{M}(k,n)$ and $x \in \mathcal{J}(\mathfrak{p})$. By definition of $A(\lambda, k, n)$, the cardinality of $\mathcal{M}$ is at least $1 + \lambda 2^{n+1}$. Let I be a cube of smallest scale in $\mathcal{M}$. Then I is contained in all cubes of $\mathcal{M}$. It follows that $I \subset A(\lambda, k, n)$. $\qquad\square$

LEMMA 5.2.5 (John Nirenberg). *For all integers* $k, n, \lambda \geq 0$, *we have*

$$\mu(A(\lambda, k, n)) \leq 2^{k+1-\lambda} \mu(G). \qquad (5.2.5)$$

PROOF. Fix k, n as in the lemma and suppress notation to write $A(\lambda)$ for $A(\lambda, k, n)$. We prove the lemma by induction on λ. For $\lambda = 0$, we use that $A(\lambda)$ by definition of $\mathfrak{M}(k,n)$ is contained in the union of elements in $\mathcal{C}(G, k)$. Lemma 5.2.2 then completes the base of the induction.

Now assume that the statement of Lemma 5.2.5 is proven for some integer $\lambda \geq 0$. The set $A(\lambda + 1)$ is contained in the set $A(\lambda)$. Let $\mathcal{M}$ be the set of dyadic cubes which are a subset of $A(\lambda)$. By Lemma 5.2.4, the union of $\mathcal{M}$ is $A(\lambda)$. Let $\mathcal{M}^*$ be the set of maximal dyadic cubes in $\mathcal{M}$.

Let $x \in A(\lambda + 1)$ and $L \in \mathcal{M}^*$ such that $x \in L$. Then by the dyadic property (2.0.8)

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n)} \mathbf{1}_{\mathcal{J}(\mathfrak{p})}(x) = \sum_{\mathfrak{p} \in \mathfrak{M}(k,n): \mathcal{J}(\mathfrak{p}) \subset L} \mathbf{1}_{\mathcal{J}(\mathfrak{p})}(x) + \sum_{\mathfrak{p} \in \mathfrak{M}(k,n): L \subsetneq \mathcal{J}(\mathfrak{p})} \mathbf{1}_{\mathcal{J}(\mathfrak{p})}(x). \qquad (5.2.6)$$

We now show

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n): \mathcal{J}(\mathfrak{p}) \subset L} \mathbf{1}_{\mathcal{J}(\mathfrak{p})}(x) \geq 2^{n+1}. \qquad (5.2.7)$$

The left-hand side of (5.2.6) is strictly greater than $(\lambda + 1)2^{n+1}$. If L is the top cube the second sum Q_2 on the right-hand side of (5.2.6) is zero and (5.2.7) follows immediately. Otherwise consider the inclusion-minimal cube $\hat{L}$ with $L \subsetneq \hat{L}$; all tiles $\mathfrak{p}$ over which Q_2 is summed over satisfy $\hat{L} \subset \mathfrak{p}$, so Q_2 is constant for all $x \in \hat{L}$. By maximality of L, Q_2 is at most $\lambda 2^{n+1}$ somewhere on $\hat{L}$, thus on all of $\hat{L}$ and consequently also at x. Rearranging the inequality yields (5.2.7).

By Lemma 5.2.3, we have

$$\sum_{\mathfrak{p}\in\mathfrak{M}(k,n):\mathcal{J}(\mathfrak{p})\subset L} \mu(E_1(\mathfrak{p})) \leq \mu(L)\,. \tag{5.2.8}$$

Multiplying by 2^n and applying (5.1.4), we obtain

$$\sum_{\mathfrak{p}\in\mathfrak{M}(k,n):\mathcal{J}(\mathfrak{p})\subset L} \mu(\mathcal{J}(\mathfrak{p})) \leq 2^n\mu(L)\,. \tag{5.2.9}$$

We then have with (5.2.7) and (5.2.9)

$$2^{n+1}\mu(A(\lambda+1)\cap L) = \int_{A(\lambda+1)\cap L} 2^{n+1} d\mu \tag{5.2.10}$$

$$\leq \int \sum_{\mathfrak{p}\in\mathfrak{M}(k,n):\mathcal{J}(\mathfrak{p})\subset L} \mathbf{1}_{\mathcal{J}(\mathfrak{p})} d\mu \leq 2^n\mu(L)\,. \tag{5.2.11}$$

Hence

$$2\mu(A(\lambda+1)) = 2\sum_{L\in\mathcal{M}^*} \mu(A(\lambda+1)\cap L) \leq \sum_{L\in\mathcal{M}^*} \mu(L) = \mu(A(\lambda))\,. \tag{5.2.12}$$

Using the induction hypothesis, this proves (5.2.5) for $\lambda+1$ and completes the proof of the lemma. □

LEMMA 5.2.6 (second exception). *We have*

$$\mu(G_2) \leq 2^{-4}\mu(G)\,. \tag{5.2.13}$$

PROOF. We use Lemma 5.2.5 and sum twice a geometric series to obtain

$$\sum_{0\leq k}\sum_{k<n} \mu(A(2n+6,k,n)) \leq \sum_{0\leq k}\sum_{k<n} 2^{k-5-2n}\mu(G) \tag{5.2.14}$$

$$\leq \sum_{0\leq k} 2^{-k-5}\mu(G) \leq 2^{-4}\mu(G)\,. \tag{5.2.15}$$

This proves the lemma. □

We turn to the set G_3.

LEMMA 5.2.7 (top tiles). *We have*

$$\sum_{\mathfrak{m}\in\mathfrak{M}(k,n)} \mu(\mathcal{J}(\mathfrak{m})) \leq 2^{n+k+3}\mu(G). \tag{5.2.16}$$

PROOF. We write the left-hand side of (5.2.16)

$$\int \sum_{\mathfrak{m}\in\mathfrak{M}(k,n)} \mathbf{1}_{\mathcal{J}(\mathfrak{m})}(x)\,d\mu(x) \leq 2^{n+1}\sum_{\lambda=0}^{|\mathfrak{M}|} \mu(A(\lambda,k,n))\,. \tag{5.2.17}$$

Using Lemma 5.2.5 and then summing a geometric series, we estimate this by

$$\leq 2^{n+1}\sum_{\lambda=0}^{|\mathfrak{M}|} 2^{k+1-\lambda}\mu(G) \leq 2^{n+1}2^{k+2}\mu(G)\,. \tag{5.2.18}$$

This proves the lemma. □

LEMMA 5.2.8 (tree count). *Let $k, n, j \geq 0$. We have for every $x \in X$*

$$\sum_{\mathfrak{u} \in \mathfrak{U}_1(k,n,j)} \mathbf{1}_{\mathcal{J}(\mathfrak{u})}(x) \leq 2^{-j} 2^{9a} \sum_{\mathfrak{m} \in \mathfrak{M}(k,n)} \mathbf{1}_{\mathcal{J}(\mathfrak{m})}(x) \qquad (5.2.19)$$

PROOF. Let $x \in X$. For each $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $x \in \mathcal{J}(\mathfrak{u})$, as $\mathfrak{u} \in \mathfrak{C}_1(k,n,j)$, there are at least 2^j elements $\mathfrak{m} \in \mathfrak{M}(k,n)$ with $100\mathfrak{u} \lesssim \mathfrak{m}$ and in particular $x \in \mathcal{J}(\mathfrak{m})$. Hence

$$\mathbf{1}_{\mathcal{J}(\mathfrak{u})}(x) \leq 2^{-j} \sum_{\mathfrak{m} \in \mathfrak{M}(k,n) : 100\mathfrak{u} \lesssim \mathfrak{m}} \mathbf{1}_{\mathcal{J}(\mathfrak{m})}(x). \qquad (5.2.20)$$

Conversely, for each $\mathfrak{m} \in \mathfrak{M}(k,n)$ with $x \in \mathcal{J}(\mathfrak{m})$, let $\mathfrak{U}(\mathfrak{m})$ be the set of $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $x \in \mathcal{J}(\mathfrak{u})$ and $100\mathfrak{u} \lesssim \mathfrak{m}$. Summing (5.2.20) over $\mathfrak{u}$ and counting the pairs $(\mathfrak{u}, \mathfrak{m})$ with $100\mathfrak{u} \lesssim \mathfrak{m}$ differently gives

$$\sum_{\mathfrak{u} \in \mathfrak{U}_1(k,n,j)} \mathbf{1}_{\mathcal{J}(\mathfrak{u})}(x) \leq 2^{-j} \sum_{\mathfrak{m} \in \mathfrak{M}(k,n)} \sum_{\mathfrak{u} \in \mathfrak{U}(\mathfrak{m})} \mathbf{1}_{\mathcal{J}(\mathfrak{m})}(x). \qquad (5.2.21)$$

We estimate the number of elements in $\mathfrak{U}(\mathfrak{m})$. Let $\mathfrak{u} \in \mathfrak{U}(\mathfrak{m})$. Then by definition of $\mathfrak{U}(\mathfrak{m})$

$$d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \mathcal{Q}(\mathfrak{m})) \leq 100. \qquad (5.2.22)$$

If $\mathfrak{u}'$ is a further element in $\mathfrak{U}(\mathfrak{m})$ with $\mathfrak{u} \neq \mathfrak{u}'$, then

$$\mathcal{Q}(\mathfrak{m}) \in B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100). \qquad (5.2.23)$$

By the last display and definition of $\mathfrak{U}_1(k,n,j)$, none of $\mathcal{J}(\mathfrak{u})$, $\mathcal{J}(\mathfrak{u}')$ is strictly contained in the other. As both contain x, we have $\mathcal{J}(\mathfrak{u}) = \mathcal{J}(\mathfrak{u}')$. We then have $d_{\mathfrak{u}} = d_{\mathfrak{u}'}$.

By (2.0.15), the balls $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 0.2)$ and $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}'), 0.2)$ are contained respectively in $\Omega(\mathfrak{u})$ and $\Omega(\mathfrak{u}')$ and thus are disjoint by (2.0.13). By (5.2.22) and the triangle inequality, both balls are contained in $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{m}), 100.2)$.

By (1.0.11) applied nine times, there is a collection of at most 2^{9a} balls of radius 0.2 with respect to the metric $d_{\mathfrak{u}}$ which cover the ball $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{m}), 100.2)$. Let B' be a ball in this cover. As the center of B' can be in at most one of the disjoint balls $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 0.2)$ and $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}'), 0.2)$, the ball B' can contain at most one of the points $\mathcal{Q}(\mathfrak{u})$, $\mathcal{Q}(\mathfrak{u}')$.

Hence the image of $\mathfrak{U}(\mathfrak{m})$ under $\mathcal{Q}$ has at most 2^{9a} elements; since $\mathcal{Q}$ is injective on $\mathfrak{U}(\mathfrak{m})$, the same is true of $\mathfrak{U}(\mathfrak{m})$. Inserting this into (5.2.21) proves the lemma. $\quad\square$

LEMMA 5.2.9 (boundary exception). *Let $\mathcal{L}(\mathfrak{u})$ be as defined in (5.1.20). We have for each $\mathfrak{u} \in \mathfrak{U}_1(k,n,l)$,*

$$\mu\Big(\bigcup_{I \in \mathcal{L}(\mathfrak{u})} I \Big) \leq D^{1-\kappa Z(n+1)} \mu(\mathcal{J}(\mathfrak{u})). \qquad (5.2.24)$$

PROOF. Let $\mathfrak{u} \in \mathfrak{U}_1(k,n,l)$. Let $I \in \mathcal{L}(\mathfrak{u})$. Then we have $s(I) = \mathrm{s}(\mathfrak{u}) - Z(n+1) - 1$ and $I \subset \mathcal{J}(\mathfrak{u})$ and $B(c(I), 8D^{s(I)}) \not\subset \mathcal{J}(\mathfrak{u})$. By (2.0.10), the set I is contained in $B(c(I), 4D^{s(I)})$. By the triangle inequality, the set I is contained in

$$X(\mathfrak{u}) := \{x \in \mathcal{J}(\mathfrak{u}) : \rho(x, X \setminus \mathcal{J}(\mathfrak{u})) \leq 12 D^{\mathrm{s}(\mathfrak{u}) - Z(n+1) - 1}\}. \qquad (5.2.25)$$

By the small boundary property (2.0.11), noting that

$$12 D^{\mathrm{s}(\mathfrak{u}) - Z(n+1) - 1} = 12 D^{s(I)} > D^{-S},$$

we have

$$\mu(X(\mathfrak{u})) \leq 2 \cdot (12D^{-Z(n+1)-1})^{\kappa} \mu(\mathcal{J}(\mathfrak{u})).$$

Using $\kappa < 1$ and $D \geq 12$, this proves the lemma. □

LEMMA 5.2.10 (third exception). *We have*

$$\mu(G_3) \leq 2^{-4}\mu(G). \tag{5.2.26}$$

PROOF. As each $\mathfrak{p} \in \mathfrak{L}_4(k,n,j)$ is contained in $\cup\mathcal{L}(\mathfrak{u})$ for some $\mathfrak{u} \in \mathfrak{U}_1(k,n,l)$, we have

$$\mu(\bigcup_{\mathfrak{p}\in\mathfrak{L}_4(k,n,j)} \mathcal{J}(\mathfrak{p})) \leq \sum_{\mathfrak{u}\in\mathfrak{U}_1(k,n,j)} \mu(\bigcup_{I\in\mathcal{L}(\mathfrak{u})} I). \tag{5.2.27}$$

Using Lemma 5.2.9 and then Lemma 5.2.8, we estimate this further by

$$\leq \sum_{\mathfrak{u}\in\mathfrak{U}_1(k,n,j)} D^{1-\kappa Z(n+1)}\mu(\mathcal{J}(\mathfrak{u})) \tag{5.2.28}$$

$$\leq 2^{100a^2+9a+1-j} \sum_{\mathfrak{m}\in\mathfrak{M}(k,n)} D^{-\kappa Z(n+1)}\mu(\mathcal{J}(\mathfrak{m})). \tag{5.2.29}$$

Using Lemma 5.2.7, we estimate this by

$$\leq 2^{100a^2+9a+1-j} D^{-\kappa Z(n+1)} 2^{n+k+3}\mu(G). \tag{5.2.30}$$

Now we estimate G_3 defined in (5.1.28) by

$$\mu(G_3) \leq \sum_{k\geq0}\sum_{n\geq k}\sum_{0\leq j\leq 2n+3} \mu(\bigcup_{\mathfrak{p}\in\mathfrak{L}_4(k,n,j)} \mathcal{J}(\mathfrak{p})) \tag{5.2.31}$$

$$\leq \sum_{k\geq0}\sum_{n\geq k}\sum_{0\leq j\leq 2n+3} 2^{100a^2+9a+4+n+k-j} D^{-\kappa Z(n+1)}\mu(G) \tag{5.2.32}$$

Summing geometric series, using that $D^{\kappa Z} \geq 8$ by (2.0.1), (2.0.2) and (2.0.3), we estimate this by

$$\leq \sum_{k\geq0}\sum_{n\geq k} 2^{100a^2+9a+5+n+k} D^{-\kappa Z(n+1)}\mu(G) \tag{5.2.33}$$

$$= \sum_{k\geq0} 2^{100a^2+9a+5+2k} D^{-\kappa Z(k+1)} \sum_{n\geq k} 2^{n-k} D^{-\kappa Z(n-k)}\mu(G) \tag{5.2.34}$$

$$\leq \sum_{k\geq0} 2^{100a^2+9a+6+2k} D^{-\kappa Z(k+1)}\mu(G) \tag{5.2.35}$$

$$\leq 2^{100a^2+9a+7} D^{-\kappa Z}\mu(G) \tag{5.2.36}$$

Using $D = 2^{100a^2}$ and $a \geq 4$ and $\kappa Z \geq 2$ by (2.0.1) and (2.0.2) proves the lemma. □

PROOF OF LEMMA 5.1.1. Adding up the bounds in Lemmas 5.2.1, 5.2.6, and 5.2.10 proves Lemma 5.1.1. □

5.3. Auxiliary lemmas

Before proving Lemma 5.1.2 and Lemma 5.1.3, we collect some useful properties of $\lesssim$.

LEMMA 5.3.1 (wiggle order 1). *If $n\mathfrak{p} \lesssim m\mathfrak{p}'$ and $n' \geq n$ and $m \geq m'$ then $n'\mathfrak{p} \lesssim m'\mathfrak{p}'$.*

PROOF. This follows immediately from the definition (2.0.24) of $\lesssim$ and the two inclusions $B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n')$ and $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m') \subset B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m)$. $\qquad\square$

LEMMA 5.3.2 (wiggle order 2). *Let $n, m \geq 1$ and $k > 0$. If $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}$ with $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{p}')$ and*

$$n\mathfrak{p} \lesssim k\mathfrak{p}' \tag{5.3.1}$$

then

$$(n + 2^{-95a}m)\mathfrak{p} \lesssim m\mathfrak{p}'. \tag{5.3.2}$$

PROOF. The assumption (5.3.1) together with the definition (2.0.24) of $\lesssim$ implies that $\mathcal{I}(\mathfrak{p}) \subsetneq \mathcal{I}(\mathfrak{p}')$. Let $\vartheta \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m)$. Then we have by the triangle inequality

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta) \leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}')) + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \vartheta)$$

The first summand is bounded by n since

$$\mathcal{Q}(\mathfrak{p}') \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), k) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n),$$

using (2.0.24). For the second summand we use Lemma 2.1.2 to show that the sum is estimated by

$$n + 2^{-95a}d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta) < n + 2^{-95a}m.$$

Thus $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n + 2^{-95a}m)$. Combined with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$, this yields (5.3.2). $\qquad\square$

LEMMA 5.3.3 (wiggle order 3). *The following implications hold for all $\mathfrak{q}, \mathfrak{q}' \in \mathfrak{P}$:*

$$\mathfrak{q} \leq \mathfrak{q}' \text{ and } \lambda \geq 1.1 \implies \lambda\mathfrak{q} \lesssim \lambda\mathfrak{q}', \tag{5.3.3}$$

$$10\mathfrak{q} \lesssim \mathfrak{q}' \text{ and } \mathcal{I}(\mathfrak{q}) \neq \mathcal{I}(\mathfrak{q}') \implies 100\mathfrak{q} \lesssim 100\mathfrak{q}', \tag{5.3.4}$$

$$2\mathfrak{q} \lesssim \mathfrak{q}' \text{ and } \mathcal{I}(\mathfrak{q}) \neq \mathcal{I}(\mathfrak{q}') \implies 4\mathfrak{q} \lesssim 500\mathfrak{q}'. \tag{5.3.5}$$

PROOF. (5.3.4) and (5.3.5) are easy consequences of Lemma 5.3.1, Lemma 5.3.2 and the fact that $a \geq 4$. For (5.3.3), if $\mathcal{I}(\mathfrak{q}) = \mathcal{I}(\mathfrak{q}')$ then we get $\mathfrak{q} = \mathfrak{q}'$ by (2.0.13) and (2.0.23). If $\mathcal{I}(\mathfrak{q}) \neq \mathcal{I}(\mathfrak{q}')$, then from (2.0.23), (2.0.24) and (2.0.15) it follows that $\mathfrak{q} \lesssim 0.2\mathfrak{q}'$, and (5.3.3) follows from an easy calculation using Lemma 5.3.2. $\qquad\square$

We call a collection $\mathfrak{A}$ of tiles convex if

$$\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}'' \text{ and } \mathfrak{p}, \mathfrak{p}'' \in \mathfrak{A} \implies \mathfrak{p}' \in \mathfrak{A}. \tag{5.3.6}$$

LEMMA 5.3.4 (P convex). *For each k, the collection $\mathfrak{P}(k)$ is convex.*

PROOF. Suppose that $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$ and $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{P}(k)$. By (5.1.3) we have $\mathcal{I}(\mathfrak{p}), \mathcal{I}(\mathfrak{p}'') \in \mathcal{C}(G, k)$, so there exists by (5.1.1) some $J \in \mathcal{D}$ with

$$\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}'') \subset J$$

and $\mu(G \cap J) > 2^{-k-1}\mu(J)$. Thus (5.1.1) holds for $\mathcal{I}(\mathfrak{p}')$. On the other hand, by (5.1.2), there exists no $J \in \mathcal{D}$ with $\mathcal{I}(\mathfrak{p}) \subset J$ and $\mu(G \cap J) > 2^{-k}\mu(J)$. Since $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$, this implies that (5.1.2) holds for $\mathcal{I}(\mathfrak{p}')$. Hence $\mathcal{I}(\mathfrak{p}') \in \mathcal{C}(G, k)$, and therefore by (5.1.3) $\mathfrak{p}' \in \mathfrak{P}(k)$. $\square$

LEMMA 5.3.5 (C convex). *For each k, n, the collection $\mathfrak{C}(k, n)$ is convex.*

PROOF. Let $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}(k, n)$. Then, in particular, $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{P}(k)$, so, by Lemma 5.3.4, $\mathfrak{p}' \in \mathfrak{P}(k)$. Next, we show that if $\mathfrak{q} \leq \mathfrak{q}' \in \mathfrak{P}(k)$ then $\mathrm{dens}'_k(\{\mathfrak{q}\}) \geq \mathrm{dens}'_k(\{\mathfrak{q}'\})$. If $\mathfrak{p} \in \mathfrak{P}(k)$ and $\lambda \geq 2$ with $\lambda\mathfrak{q}' \lesssim \lambda\mathfrak{p}$, then it follows from $\mathfrak{q} \leq \mathfrak{q}'$, (5.3.3) of Lemma 5.3.3 and transitivity of $\lesssim$ that $\lambda\mathfrak{q} \lesssim \lambda\mathfrak{p}$. Thus the supremum in the definition (5.1.6) of $\mathrm{dens}'_k(\{\mathfrak{q}\})$ is over a superset of the set the supremum in the definition of $\mathrm{dens}'_k(\{\mathfrak{q}'\})$ is taken over, which shows $\mathrm{dens}'_k(\{\mathfrak{q}\}) \geq \mathrm{dens}'_k(\{\mathfrak{q}'\})$. From $\mathfrak{p}' \leq \mathfrak{p}''$, $\mathfrak{p}'' \in \mathfrak{C}(k, n)$ and (5.1.7) it then follows that

$$2^{4a}2^{-n} < \mathrm{dens}'_k(\{\mathfrak{p}''\}) \leq \mathrm{dens}'_k(\{\mathfrak{p}'\}) .$$

Similarly, it follows from $\mathfrak{p} \leq \mathfrak{p}'$, $\mathfrak{p} \in \mathfrak{C}(k, n)$ and (5.1.7) that

$$\mathrm{dens}'_k(\{\mathfrak{p}'\}) \leq \mathrm{dens}'_k(\{\mathfrak{p}\}) \leq 2^{4a}2^{-n+1} .$$

Thus $\mathfrak{p}' \in \mathfrak{C}(k, n)$. $\square$

LEMMA 5.3.6 (C1 convex). *For each k, n, j, the collection $\mathfrak{C}_1(k, n, j)$ is convex.*

PROOF. Let $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_1(k, n, j)$. By Lemma 5.3.5 and the inclusion $\mathfrak{C}_1(k, n, j) \subset \mathfrak{C}(k, n)$, which holds by definition (5.1.9), we have $\mathfrak{p}' \in \mathfrak{C}(k, n)$. By (5.3.3) and transitivity of $\lesssim$ we have that $\mathfrak{q} \leq \mathfrak{q}'$ and $100\mathfrak{q}' \lesssim \mathfrak{m}$ imply $100\mathfrak{q} \lesssim \mathfrak{m}$. So, by (5.1.8), $\mathfrak{B}(\mathfrak{p}'') \subset \mathfrak{B}(\mathfrak{p}') \subset \mathfrak{B}(\mathfrak{p})$. Consequently, by (5.1.9)

$$2^j \leq |\mathfrak{B}(\mathfrak{p}'')| \leq |\mathfrak{B}(\mathfrak{p}')| \leq |\mathfrak{B}(\mathfrak{p})| < 2^{j+1} ,$$

thus $\mathfrak{p}' \in \mathfrak{C}_1(k, n, j)$. $\square$

LEMMA 5.3.7 (C2 convex). *For each k, n, j, the collection $\mathfrak{C}_2(k, n, j)$ is convex.*

PROOF. Let $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_2(k, n, j)$. By (5.1.13), we have

$$\mathfrak{C}_2(k, n, j) \subset \mathfrak{C}_1(k, n, j) .$$

Combined with Lemma 5.3.6, it follows that $\mathfrak{p}' \in \mathfrak{C}_1(k, n, j)$. If $\mathfrak{p} = \mathfrak{p}'$ the statement is trivially true, otherwise suppose that $\mathfrak{p}' \notin \mathfrak{C}_2(k, n, j)$. By (5.1.13), this implies that there exists $0 \leq l' \leq Z(n+1)$ with $\mathfrak{p}' \in \mathfrak{L}_1(k, n, j, l')$. By the definition (5.1.11) of $\mathfrak{L}_1(k, n, j, l')$, this implies that $\mathfrak{p}'$ is minimal with respect to $\leq$ in $\mathfrak{C}_1(k, n, j) \smallsetminus \bigcup_{l<l'} \mathfrak{L}_1(k, n, j, l)$. Since $\mathfrak{p} \leq \mathfrak{p}'$ and $\mathfrak{p} \in \mathfrak{C}_2(k, n, j)$, $\mathfrak{p} = \mathfrak{p}'$, a contradiction. $\square$

LEMMA 5.3.8 (C3 convex). *For each k, n, j, the collection $\mathfrak{C}_3(k, n, j)$ is convex.*

PROOF. Let $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_3(k, n, j)$. By (5.1.16) and Lemma 5.3.7 it follows that $\mathfrak{p}' \in \mathfrak{C}_2(k, n, j)$. By (5.1.16) and (5.1.15), there exists $\mathfrak{u} \in \mathfrak{U}_1(k, n, j)$ with $2\mathfrak{p}'' \lesssim \mathfrak{u}$ and $\mathcal{I}(\mathfrak{p}'') \subsetneq \mathcal{I}(\mathfrak{u})$. From $\mathfrak{p}' \leq \mathfrak{p}''$, (5.3.3) and transitivity of $\lesssim$ we then have $2\mathfrak{p}' \lesssim \mathfrak{u}$ and $\mathcal{I}(\mathfrak{p}') \subsetneq \mathcal{I}(\mathfrak{u})$, so $\mathfrak{p}' \in \mathfrak{C}_3(k, n, j)$. $\square$

LEMMA 5.3.9 (C4 convex). *For each k, n, j, the collection $\mathfrak{C}_4(k, n, j)$ is convex.*

PROOF. The proof is entirely analogous to Lemma 5.3.7, substituting $\mathfrak{C}_4$ for $\mathfrak{C}_2$, $\mathfrak{C}_3$ for $\mathfrak{C}_1$ and $\mathfrak{p}' \leq \mathfrak{p}''$ for $\mathfrak{p} \leq \mathfrak{p}'$. $\square$

LEMMA 5.3.10 (C5 convex). *For each k, n, j, the collection $\mathfrak{C}_5(k, n, j)$ is convex.*

PROOF. Let $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_5(k, n, j)$. Then $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_4(k, n, j)$ by (5.1.23), and thus by Lemma 5.3.9 $\mathfrak{p}' \in \mathfrak{C}_4(k, n, j)$. It suffices to show that if $\mathfrak{p}' \in \mathfrak{L}_4(k, n, j)$ then $\mathfrak{p} \in \mathfrak{L}_4(k, n, j)$ by contraposition; this is true by (5.1.22) and $\mathfrak{p} \leq \mathfrak{p}'$. $\square$

LEMMA 5.3.11 (dens compare). *We have for every $k \geq 0$ and $\mathfrak{P}' \subset \mathfrak{P}(k)$*

$$\mathrm{dens}_1(\mathfrak{P}') \leq \mathrm{dens}_k'(\mathfrak{P}') . \tag{5.3.7}$$

PROOF. It suffices to show that for all $\mathfrak{p}' \in \mathfrak{P}'$ and $\lambda \geq 2$ and $\mathfrak{p} \in \mathfrak{P}(\mathfrak{P}')$ with $\lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}$ we have

$$\frac{\mu(E_2(\lambda, \mathfrak{p}))}{\mu(\mathcal{I}(\mathfrak{p}))} \leq \sup_{\mathfrak{p}'' \in \mathfrak{P}(k): \lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}''} \frac{\mu(E_2(\lambda, \mathfrak{p}''))}{\mu(\mathcal{I}(\mathfrak{p}''))}. \tag{5.3.8}$$

Let such $\mathfrak{p}'$, λ, $\mathfrak{p}$ be given. It suffices to show that $\mathfrak{p} \in \mathfrak{P}(k)$, that is, it satisfies (5.1.1) and (5.1.2).

We show (5.1.1). As $\mathfrak{p} \in \mathfrak{P}(\mathfrak{P}')$, there exists $\mathfrak{p}'' \in \mathfrak{P}'$ with $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}'')$. By assumption on $\mathfrak{P}'$, we have $\mathfrak{p}'' \in \mathfrak{P}(k)$ and there exists $J \in \mathcal{D}$ with $\mathcal{I}(\mathfrak{p}'') \subset J$ and

$$\mu(G \cap J) > 2^{-k-1}\mu(J). \tag{5.3.9}$$

Then also $\mathcal{I}(\mathfrak{p}') \subset J$, which proves (5.1.1) for $\mathfrak{p}$.

We show (5.1.2). Assume to get a contradiction that there exists $J \in \mathcal{D}$ with $\mathcal{I}(\mathfrak{p}) \subset J$ and

$$\mu(G \cap J) > 2^{-k}\mu(J). \tag{5.3.10}$$

As $\lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}$, we have $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p})$, and therefore $\mathcal{I}(\mathfrak{p}') \subset J$. This contradicts $\mathfrak{p}' \in \mathfrak{P}' \subset \mathfrak{P}(k)$. This proves (5.1.2) for $\mathfrak{p}$. $\square$

LEMMA 5.3.12 (C dens1). *For each set $\mathfrak{A} \subset \mathfrak{C}(k, n)$, we have*

$$\mathrm{dens}_1(\mathfrak{A}) \leq 2^{4a}2^{-n+1} .$$

PROOF. We have by Lemma 5.3.11 that $\mathrm{dens}_1(\mathfrak{A}) \leq \mathrm{dens}_k'(\mathfrak{A})$. Since $\mathfrak{A} \subset \mathfrak{C}(k, n)$, it follows from monotonicity of suprema and the definition (5.1.6) that $\mathrm{dens}_k'(\mathfrak{A}) \leq \mathrm{dens}_k'(\mathfrak{C}(k, n))$. By (5.1.6) and (5.1.7), we have

$$\mathrm{dens}_k'(\mathfrak{C}(k, n)) = \sup_{\mathfrak{p} \in \mathfrak{C}(k,n)} \mathrm{dens}_k'(\{\mathfrak{p}\}) \leq 2^{4a}2^{-n+1} .$$

$\square$

5.4. Proof of the Forest Union Lemma

Fix $k, n, j \geq 0$. Define
$$\mathfrak{C}_6(k, n, j)$$
to be the set of all tiles $\mathfrak{p} \in \mathfrak{C}_5(k, n, j)$ such that $\mathcal{I}(\mathfrak{p}) \not\subset G'$. The following chain of lemmas establishes that the set $\mathfrak{C}_6(k, n, j)$ can be written as a union of a small number of n-forests.

For $\mathfrak{u} \in \mathfrak{U}_1(k, n, j)$, define
$$\mathfrak{T}_1(\mathfrak{u}) := \{\mathfrak{p} \in \mathfrak{C}_1(k, n, j) \ : \mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u}), \ 2\mathfrak{p} \lesssim \mathfrak{u}\} . \tag{5.4.1}$$

Define
$$\mathfrak{U}_2(k, n, j) := \{\mathfrak{u} \in \mathfrak{U}_1(k, n, j) : \mathfrak{T}_1(\mathfrak{u}) \cap \mathfrak{C}_6(k, n, j) \neq \varnothing\} . \tag{5.4.2}$$

Define a relation $\sim$ on $\mathfrak{U}_2(k, n, j)$ by setting $\mathfrak{u} \sim \mathfrak{u}'$ for $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}_2(k, n, j)$ if $\mathfrak{u} = \mathfrak{u}'$ or there exists $\mathfrak{p}$ in $\mathfrak{T}_1(\mathfrak{u})$ with $10\mathfrak{p} \lesssim \mathfrak{u}'$.

LEMMA 5.4.1 (relation geometry). *If $\mathfrak{u} \sim \mathfrak{u}'$, then $\mathcal{I}(u) = \mathcal{I}(u')$ and*
$$B_\mathfrak{u}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) \neq \varnothing .$$

PROOF. Let $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}_2(k, n, j)$ with $\mathfrak{u} \sim \mathfrak{u}'$. If $\mathfrak{u} = \mathfrak{u}'$ then the conclusion of the Lemma clearly holds. Else, there exists $\mathfrak{p} \in \mathfrak{C}_1(k, n, j)$ such that $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u})$ and $2\mathfrak{p} \lesssim \mathfrak{u}$ and $10\mathfrak{p} \lesssim \mathfrak{u}'$. Using Lemma 5.3.1 and (5.3.4) of Lemma 5.3.3, we deduce that
$$100\mathfrak{p} \lesssim 100\mathfrak{u} , \qquad 100\mathfrak{p} \lesssim 100\mathfrak{u}' . \tag{5.4.3}$$
Now suppose that $B_\mathfrak{u}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) = \varnothing$. Then we have $\mathfrak{B}(\mathfrak{u}) \cap \mathfrak{B}(\mathfrak{u}') = \varnothing$, by the definition (5.1.8) of $\mathfrak{B}$ and the definition (2.0.24) of $\lesssim$, but also $\mathfrak{B}(\mathfrak{u}) \subset \mathfrak{B}(\mathfrak{p})$ and $\mathfrak{B}(\mathfrak{u}') \subset \mathfrak{B}(\mathfrak{p})$, by (5.1.8), (2.0.24) and (5.4.3). Hence,
$$|\mathfrak{B}(\mathfrak{p})| \geq |\mathfrak{B}(\mathfrak{u})| + |\mathfrak{B}(\mathfrak{u}')| \geq 2^j + 2^j = 2^{j+1} ,$$
which contradicts $\mathfrak{p} \in \mathfrak{C}_1(k, n, j)$. Therefore we must have
$$B_\mathfrak{u}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) \neq \varnothing .$$

It follows from $2\mathfrak{p} \lesssim \mathfrak{u}$ and $10\mathfrak{p} \lesssim \mathfrak{u}'$ that $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u})$ and $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}')$. By (2.0.8), it follows that $\mathcal{I}(\mathfrak{u})$ and $\mathcal{I}(\mathfrak{u}')$ are nested. Combining this with the conclusion of the last paragraph and definition (5.1.14) of $\mathfrak{U}_1(k, n, j)$, we obtain that $\mathcal{I}(\mathfrak{u}) = \mathcal{I}(\mathfrak{u}')$. $\qquad\square$

LEMMA 5.4.2 (equivalence relation). *For each k, n, j, the relation $\sim$ on $\mathfrak{U}_2(k, n, j)$ is an equivalence relation.*

PROOF. Reflexivity holds by definition. For transitivity, suppose that
$$\mathfrak{u}, \mathfrak{u}', \mathfrak{u}'' \in \mathfrak{U}_1(k, n, j)$$
and $\mathfrak{u} \sim \mathfrak{u}'$, $\mathfrak{u}' \sim \mathfrak{u}''$. By Lemma 5.4.1, it follows that $\mathcal{I}(\mathfrak{u}) = \mathcal{I}(\mathfrak{u}') = \mathcal{I}(\mathfrak{u}'')$, that there exists
$$\vartheta \in B_\mathfrak{u}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100)$$
and that there exists
$$\theta \in B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) \cap B_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), 100) .$$

If $\mathfrak{u} = \mathfrak{u}'$, then $\mathfrak{u} \sim \mathfrak{u}''$ holds by assumption. Else, there exists by the definition of $\sim$ some $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u})$ with $10\mathfrak{p} \lesssim \mathfrak{u}'$. Then we have $2\mathfrak{p} \lesssim \mathfrak{u}$ and $\mathfrak{p} \neq \mathfrak{u}$ by definition of $\mathfrak{T}_1(\mathfrak{u})$, so $4\mathfrak{p} \lesssim 500\mathfrak{u}$ by (5.3.5). For $q \in B_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), 1)$ it follows by the triangle inequality that

$$d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), q) \leq d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \vartheta) + d_{\mathfrak{u}}(\vartheta, \mathcal{Q}(\mathfrak{u}'))$$
$$+ d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}'), \theta) + d_{\mathfrak{u}}(\theta, \mathcal{Q}(\mathfrak{u}'')) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}''), q).$$

Using (2.0.17) and the fact that $\mathcal{J}(\mathfrak{u}) = \mathcal{J}(\mathfrak{u}') = \mathcal{J}(\mathfrak{u}'')$ this equals

$$d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \vartheta) + d_{\mathfrak{u}'}(\vartheta, \mathcal{Q}(\mathfrak{u}'))$$
$$+ d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), \theta) + d_{\mathfrak{u}''}(\theta, \mathcal{Q}(\mathfrak{u}'')) + d_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), q)$$
$$< 100 + 100 + 100 + 100 + 1 < 500.$$

Since $4\mathfrak{p} \lesssim 500\mathfrak{u}$, it follows that $d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), q) < 4 < 10$. We have shown that $B_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), 1) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 10)$, combining this with $\mathcal{J}(\mathfrak{u}'') = \mathcal{J}(\mathfrak{u})$ gives $\mathfrak{u} \sim \mathfrak{u}''$.

For symmetry suppose that $\mathfrak{u} \sim \mathfrak{u}'$. By Lemma (5.4.1), it follows that $\mathcal{J}(\mathfrak{u}) = \mathcal{J}(\mathfrak{u}')$ and that there exists $\vartheta \in B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100)$. Again, for $\mathfrak{u} = \mathfrak{u}'$ symmetry is obvious, so suppose that $\mathfrak{u} \neq \mathfrak{u}'$. There exists $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u}')$, which then satisfies $2\mathfrak{p} \lesssim \mathfrak{u}'$ and $\mathcal{J}(\mathfrak{p}) \neq \mathcal{J}(\mathfrak{u}')$. By Lemma 5.3.1 and (5.3.5), it follows that

$$10\mathfrak{p} \lesssim 4\mathfrak{p} \lesssim 500\mathfrak{u}'. \tag{5.4.4}$$

If $q \in B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 1)$ then we have from the triangle inequality and the fact that $\mathcal{J}(\mathfrak{u}) = \mathcal{J}(\mathfrak{u}')$:

$$d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), q) \leq d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), \vartheta) + d_{\mathfrak{u}'}(\vartheta, \mathcal{Q}(\mathfrak{u})) + d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}), q)$$
$$= d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), \vartheta) + d_{\mathfrak{u}}(\vartheta, \mathcal{Q}(\mathfrak{u})) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), q)$$
$$< 100 + 100 + 1 < 500.$$

Combining this with (5.4.4) and (2.0.24), we get

$$B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 1) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 10).$$

Since $2\mathfrak{p} \lesssim \mathfrak{u}'$, we have $\mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u}') = \mathcal{J}(\mathfrak{u})$. Thus, $10\mathfrak{p} \lesssim \mathfrak{u}$ which completes the proof of $\mathfrak{u}' \sim \mathfrak{u}$. $\qquad\square$

Choose a set $\mathfrak{U}_3(k, n, j)$ of representatives for the equivalence classes of $\sim$ in $\mathfrak{U}_2(k, n, j)$. Define for each $\mathfrak{u} \in \mathfrak{U}_3(k, n, j)$

$$\mathfrak{T}_2(\mathfrak{u}) := \bigcup_{\mathfrak{u} \sim \mathfrak{u}'} \mathfrak{T}_1(\mathfrak{u}') \cap \mathfrak{C}_6(k, n, j). \tag{5.4.5}$$

LEMMA 5.4.3 (C6 forest). *We have*

$$\mathfrak{C}_6(k, n, j) = \bigcup_{\mathfrak{u} \in \mathfrak{U}_3(k, n, j)} \mathfrak{T}_2(\mathfrak{u}). \tag{5.4.6}$$

PROOF. Let $\mathfrak{p} \in \mathfrak{C}_6(k, n, j)$. By (5.1.19) and (5.1.23), we have $\mathfrak{p} \in \mathfrak{C}_3(k, n, j)$. By (5.1.15) and (5.1.16), there exists $\mathfrak{u} \in \mathfrak{U}_1(k, n, j)$ with $2\mathfrak{p} \lesssim \mathfrak{u}$ and $\mathcal{J}(\mathfrak{p}) \neq \mathcal{J}(\mathfrak{u})$, that is, with $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u})$. Then $\mathfrak{T}_1(\mathfrak{u})$ is clearly nonempty, so $\mathfrak{u} \in \mathfrak{U}_2(k, n, j)$. By the definition of $\mathfrak{U}_3(k, n, j)$, there exists $\mathfrak{u}' \in \mathfrak{U}_3(k, n, j)$ with $\mathfrak{u} \sim \mathfrak{u}'$. By (5.4.5), we have $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u}')$. $\qquad\square$

LEMMA 5.4.4 (forest geometry). *For each* $\mathfrak{u} \in \mathfrak{U}_3(k, n, j)$, *the set* $\mathfrak{T}_2(\mathfrak{u})$ *satisfies* (2.0.32).

PROOF. Let $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$. By (5.4.5), there exists $\mathfrak{u}' \sim \mathfrak{u}$ with $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u}')$. Then we have $2\mathfrak{p} \lesssim \mathfrak{u}'$ and $\mathcal{J}(\mathfrak{p}) \neq \mathcal{J}(\mathfrak{u}')$, so by (5.3.5) $4\mathfrak{p} \lesssim 500\mathfrak{u}'$. Further, by Lemma 5.4.1, we have that $\mathcal{J}(\mathfrak{u}') = \mathcal{J}(\mathfrak{u})$ and there exists $\vartheta \in B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) \cap B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100)$. Let $\theta \in B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 1)$. Using the triangle inequality and the fact that $\mathcal{J}(\mathfrak{u}') = \mathcal{J}(\mathfrak{u})$, we obtain

$$
\begin{aligned}
d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), \theta) &\leq d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), \vartheta) + d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}), \vartheta) + d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}), \theta) \\
&= d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), \vartheta) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \vartheta) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \theta) \\
&< 100 + 100 + 1 < 500 \,.
\end{aligned}
$$

Combining this with $4\mathfrak{p} \lesssim 500\mathfrak{u}'$, we obtain

$$
B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 1) \subset B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 500) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 4) \,.
$$

Together with $\mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u}') = \mathcal{J}(\mathfrak{u})$, this gives $4\mathfrak{p} \lesssim 1\mathfrak{u}$, which is (2.0.32). $\qquad \square$

LEMMA 5.4.5 (forest convex). *For each* $\mathfrak{u} \in \mathfrak{U}_3(k, n, j)$, *the set* $\mathfrak{T}_2(\mathfrak{u})$ *satisfies the convexity condition* (2.0.33).

PROOF. Let $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{T}_2(\mathfrak{u})$ and $\mathfrak{p}' \in \mathfrak{P}$ with $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$. By (5.4.5) we have $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_6(k, n, j) \subset \mathfrak{C}_5(k, n, j)$. By Lemma 5.3.10, we have $\mathfrak{p}' \in \mathfrak{C}_5(k, n, j)$. Since $\mathfrak{p} \in \mathfrak{C}_6(k, n, j)$ we have $\mathcal{J}(\mathfrak{p}) \not\subset G'$, so $\mathcal{J}(\mathfrak{p}') \not\subset G'$ and therefore also $\mathfrak{p}' \in \mathfrak{C}_6(k, n, j)$.

By (5.4.5) there exists $\mathfrak{u}' \in \mathfrak{U}_2(k, n, j)$ with $\mathfrak{p}'' \in \mathfrak{T}_1(\mathfrak{u}')$ and hence $2\mathfrak{p}'' \lesssim \mathfrak{u}'$ and $\mathcal{J}(\mathfrak{p}'') \neq \mathcal{J}(\mathfrak{u}')$. Together this implies $\mathcal{J}(\mathfrak{p}'') \subsetneq \mathcal{J}(\mathfrak{u}')$. With the inclusion $\mathcal{J}(\mathfrak{p}') \subset \mathcal{J}(\mathfrak{p}'')$ from $\mathfrak{p}' \leq \mathfrak{p}''$, it follows that $\mathcal{J}(\mathfrak{p}') \subsetneq \mathcal{J}(\mathfrak{u}')$ and hence $\mathcal{J}(\mathfrak{p}') \neq \mathcal{J}(\mathfrak{u}')$. By (5.3.3) and transitivity of $\lesssim$ we further have $2\mathfrak{p}' \lesssim \mathfrak{u}'$, so $\mathfrak{p}' \in \mathfrak{T}_1(\mathfrak{u}')$. It follows that $\mathfrak{p}' \in \mathfrak{T}_2(\mathfrak{u})$, which shows (2.0.33). $\qquad \square$

LEMMA 5.4.6 (forest separation). *For each* $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}_3(k, n, j)$ *with* $\mathfrak{u} \neq \mathfrak{u}'$ *and each* $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$ *with* $\mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u}')$ *we have*

$$
d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) > 2^{Z(n+1)} \,. \tag{5.4.7}
$$

PROOF. By the definition (5.1.13) of $\mathfrak{C}_2(k, n, j)$, there exists a tile $\mathfrak{p}' \in \mathfrak{C}_1(k, n, j)$ with $\mathfrak{p}' \leq \mathfrak{p}$ and $\mathrm{s}(\mathfrak{p}') \leq \mathrm{s}(\mathfrak{p}) - Z(n+1)$. By Lemma 2.1.2 we have

$$
d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) \geq 2^{95aZ(n+1)} d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) \,.
$$

By (5.3.3) we have $2\mathfrak{p}' \lesssim 2\mathfrak{p}$, so by transitivity of $\lesssim$ there exists $\mathfrak{v} \sim \mathfrak{u}$ with $2\mathfrak{p}' \lesssim \mathfrak{v}$ and $\mathcal{J}(\mathfrak{p}') \neq \mathcal{J}(\mathfrak{v})$. Since $\mathfrak{u}, \mathfrak{u}'$ are not equivalent under $\sim$, we have $\mathfrak{v} \nsim \mathfrak{u}'$, thus $10\mathfrak{p}' \not\lesssim \mathfrak{u}'$. This implies that there exists $q \in B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 1) \setminus B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 10)$.

From $\mathfrak{p}' \leq \mathfrak{p}$, $\mathcal{J}(\mathfrak{p}') \subset \mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u}')$ and Lemma 2.1.2 it then follows that

$$
\begin{aligned}
&d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) \\
&\geq -d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}')) + d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), q) - d_{\mathfrak{p}'}(q, \mathcal{Q}(\mathfrak{u}')) \\
&\geq -d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}')) + d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), q) - d_{\mathfrak{u}'}(q, \mathcal{Q}(\mathfrak{u}')) \\
&> -1 + 10 - 1 = 8 \,.
\end{aligned}
$$

The lemma follows by combining the two displays with the fact that $95a \geq 1$. $\qquad \square$

LEMMA 5.4.7 (forest inner). *For each $\mathfrak{u} \in \mathfrak{U}_3(k, n, j)$ and each $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$ we have*

$$B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})}) \subset \mathcal{J}(\mathfrak{u}). \tag{5.4.8}$$

PROOF. Let $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$. Let

$$\mathfrak{q} \in \bigcup_{\mathfrak{u} \sim \mathfrak{u}'} \mathfrak{T}_1(\mathfrak{u}') \cap \mathfrak{C}_3(k, n, j) \tag{5.4.9}$$

be a maximal element of this set with respect to $\leq$ such that $\mathfrak{p} \leq \mathfrak{q}$. We show that there is no $\mathfrak{q}' \in \mathfrak{C}_3(k, n, j)$ with $\mathfrak{q} \leq \mathfrak{q}'$ and $\mathfrak{q} \neq \mathfrak{q}'$. Indeed, suppose $\mathfrak{q}'$ was such a tile. By (5.1.16) there exists $\mathfrak{u}'' \in \mathfrak{U}_1(k, n, j)$ with $2\mathfrak{q}' \lesssim \mathfrak{u}''$. Then we have in particular by Lemma 5.3.1 that $10\mathfrak{p} \lesssim \mathfrak{u}''$. Let $\mathfrak{u}' \sim \mathfrak{u}$ be such that $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u}')$. By definition of $\sim$, we have $\mathfrak{u}' \sim \mathfrak{u}''$, hence $\mathfrak{u} \sim \mathfrak{u}''$. This implies that $\mathfrak{q}'$ is in the set in (5.4.9), contradicting maximality of $\mathfrak{q}$.

Let $\mathfrak{u}' \sim \mathfrak{u}$ with $\mathfrak{q} \in \mathfrak{T}_1(\mathfrak{u}')$. By the definition (5.4.1) of $\mathfrak{T}_1$, we have $\mathrm{s}(\mathfrak{p}) < \mathrm{s}(\mathfrak{u}')$. By Lemma 5.4.1, we have $\mathrm{s}(\mathfrak{u}) = \mathrm{s}(\mathfrak{u}')$, hence $\mathrm{s}(\mathfrak{q}) < \mathrm{s}(\mathfrak{u})$. By definition of $\mathfrak{C}_4(k, n, j)$, $\mathfrak{p}$ is not in any of the maximal $Z(n + 1)$ layers of tiles in $\mathfrak{C}_3(k, n, j)$, and hence $\mathrm{s}(\mathfrak{p}) \leq \mathrm{s}(\mathfrak{q}) - Z(n + 1) \leq \mathrm{s}(\mathfrak{u}) - Z(n + 1) - 1$.

Thus, there exists some cube $I \in \mathcal{D}$ with $s(I) = \mathrm{s}(\mathfrak{u}) - Z(n+1) - 1$ and $I \subset \mathcal{J}(\mathfrak{u})$ and $\mathcal{J}(\mathfrak{p}) \subset I$. Since $\mathfrak{p} \in \mathfrak{C}_5(k, n, j)$, we have that $I \notin \mathcal{L}(\mathfrak{u})$, so $B(c(I), 8D^{s(I)}) \subset \mathcal{J}(\mathfrak{u})$. By the triangle inequality, (2.0.1) and $a \geq 4$, the same then holds for the subcube $\mathcal{J}(\mathfrak{p}) \subset I$. $\qquad\square$

LEMMA 5.4.8 (forest stacking). *It holds for $k < n$ that*

$$\sum_{\mathfrak{u} \in \mathfrak{U}_3(k, n, j)} \mathbf{1}_{\mathcal{J}(\mathfrak{u})} \leq (4n + 12)2^n. \tag{5.4.10}$$

PROOF. Suppose that a point x is contained in more than $(4n + 12)2^n$ cubes $\mathcal{J}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U}_3(k, n, j)$. Since $\mathfrak{U}_3(k, n, j) \subset \mathfrak{C}_1(k, n, j)$ for each such $\mathfrak{u}$, there exists $\mathfrak{m} \in \mathfrak{M}(k, n)$ such that $100\mathfrak{u} \lesssim \mathfrak{m}$. We fix such an $\mathfrak{m}(\mathfrak{u}) := \mathfrak{m}$ for each $\mathfrak{u}$, and claim that the map $\mathfrak{u} \mapsto \mathfrak{m}(\mathfrak{u})$ is injective. Indeed, assume for $\mathfrak{u} \neq \mathfrak{u}'$ there is $\mathfrak{m} \in \mathfrak{M}(k, n)$ such that $100\mathfrak{u} \lesssim \mathfrak{m}$ and $100\mathfrak{u}' \lesssim \mathfrak{m}$. By (2.0.8), either $\mathcal{J}(\mathfrak{u}) \subset \mathcal{J}(\mathfrak{u}')$ or $\mathcal{J}(\mathfrak{u}') \subset \mathcal{J}(\mathfrak{u})$. By (5.1.14), $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) = \varnothing$. This contradicts $\Omega(\mathfrak{m})$ being contained in both sets by (2.0.15). Thus x is contained in more than $(4n + 12)2^n$ cubes $\mathcal{J}(\mathfrak{m})$, $\mathfrak{m} \in \mathfrak{M}(k, n)$. Consequently, we have by (5.1.26) that $x \in A(2n + 6, k, n) \subset G_2$. Let $\mathcal{J}(\mathfrak{u})$ be an inclusion minimal cube among the $\mathcal{J}(\mathfrak{u}')$, $\mathfrak{u}' \in \mathfrak{U}_3(k, n, j)$ with $x \in \mathcal{J}(\mathfrak{u})$. By the dyadic property (2.0.8), we have $\mathcal{J}(\mathfrak{u}) \subset \mathcal{J}(\mathfrak{u}')$ for all cubes $\mathcal{J}(\mathfrak{u}')$ containing x. Thus

$$\mathcal{J}(\mathfrak{u}) \subset \{y : \sum_{\mathfrak{u} \in \mathfrak{U}_3(k, n, j)} \mathbf{1}_{\mathcal{J}(\mathfrak{u})}(y) > 1 + (4n + 12)2^n\} \subset G_2.$$

Thus $\mathfrak{T}_1(\mathfrak{u}) \cap \mathfrak{C}_6(k, n, j) = \varnothing$. This contradicts $\mathfrak{u} \in \mathfrak{U}_2(k, n, j)$. $\qquad\square$

We now turn to the proof of Lemma 5.1.2.

PROOF OF LEMMA 5.1.2. We first fix k, n, j. By (2.0.21) and (2.0.20), we have that $\mathbf{1}_{\mathcal{J}(\mathfrak{p})} T_{\mathfrak{p}} f(x) = T_{\mathfrak{p}} f(x)$ and hence $\mathbf{1}_{G \smallsetminus G'} T_{\mathfrak{p}} f(x) = 0$ for all $\mathfrak{p} \in \mathfrak{C}_5(k, n, j) \smallsetminus \mathfrak{C}_6(k, n, j)$. Thus it suffices to estimate the contribution of the sets $\mathfrak{C}_6(k, n, j)$. By

Lemma 5.4.8, we can decompose $\mathfrak{U}_3(k, n, j)$ as a disjoint union of at most $4n + 13$ collections $\mathfrak{U}_4(k, n, j, l)$, $1 \leq l \leq 4n + 13$, each satisfying

$$\sum_{\mathfrak{u} \in \mathfrak{U}_4(k,n,j,l)} \mathbf{1}_{\mathcal{J}(\mathfrak{u})} \leq 2^n \, .$$

By Lemmas 5.4.4, 5.4.5, 5.4.6, 5.4.7 and 5.3.12, the pairs

$$(\mathfrak{U}_4(k, n, j, l), \mathfrak{T}_2|_{\mathfrak{U}_4(k,n,j,l)})$$

are n-forests for each k, n, j, l, and by Lemma 5.4.3, we have

$$\mathfrak{C}_6(k, n, j) = \bigcup_{l=1}^{4n+13} \bigcup_{\mathfrak{u} \in \mathfrak{U}_4(k,n,j,l)} \mathfrak{T}_2(\mathfrak{u}) \, .$$

Since $\mathcal{J}(\mathfrak{p}) \not\subset G_1$ for all $\mathfrak{p} \in \mathfrak{C}_6(k, n, j)$, we have $\mathfrak{C}_6(k, n, j) \cap \mathfrak{P}_{F,G} = \varnothing$ and hence

$$\text{dens}_2(\bigcup_{\mathfrak{u} \in \mathfrak{U}_4(k,n,j,l)} \mathfrak{T}_2(\mathfrak{u})) \leq 2^{2a+5} \frac{\mu(F)}{\mu(G)} \, .$$

Using the triangle inequality according to the splitting by k, n, j and l in (5.1.31) and applying Proposition 2.0.4 to each term, we obtain the estimate

$$\sum_{k \geq 0} \sum_{n \geq k} (2n + 3)(4n + 13) 2^{432a^3} 2^{-(1-\frac{1}{q})n} (2^{2a+5} \frac{\mu(F)}{\mu(G)})^{\frac{1}{q}-\frac{1}{2}} \|f\|_2 \|\mathbf{1}_{G \setminus G'}\|_2$$

for the left hand side of (5.1.31). Since $|f| \leq \mathbf{1}_F$, we have $\|f\|_2 \leq \mu(F)^{1/2}$, and we have $\|\mathbf{1}_{G \setminus G'}\|_2 \leq \mu(G)^{1/2}$. Combining this with $a \geq 4$, we estimate by

$$2^{433a^3} \mu(F)^{\frac{1}{q}} \mu(G)^{1-\frac{1}{q}} \sum_{k \geq 0} \sum_{n \geq k} n^2 2^{-(1-\frac{1}{q})n} \, .$$

Interchanging the order of summation, the sum equals

$$\sum_{n \geq 0} n^2(n + 1) 2^{-\frac{q-1}{q}n} \leq \frac{2^{2a^3}}{(q-1)^4} \, ,$$

which completes the proof of the lemma. $\square$

5.5. Proof of the Forest Complement Lemma

Define $\mathfrak{P}_{G \setminus G'}$ to be the set of all $\mathfrak{p} \in \mathfrak{P}$ such that $\mu(\mathcal{J}(\mathfrak{p}) \cap (G \setminus G')) > 0$.

LEMMA 5.5.1 (antichain decomposition). *We have that*

$$\mathfrak{P}_2 \cap \mathfrak{P}_{G \smallsetminus G'} \tag{5.5.1}$$

$$= \bigcup_{k \geq 0} \bigcup_{n \geq k} \mathfrak{L}_0(k, n) \cap \mathfrak{P}_{G \smallsetminus G'} \tag{5.5.2}$$

$$\cup \bigcup_{k \geq 0} \bigcup_{n \geq k} \bigcup_{0 \leq j \leq 2n+3} \mathfrak{L}_2(k, n, j) \cap \mathfrak{P}_{G \smallsetminus G'} \tag{5.5.3}$$

$$\cup \bigcup_{k \geq 0} \bigcup_{n \geq k} \bigcup_{0 \leq j \leq 2n+3} \bigcup_{0 \leq l \leq Z(n+1)} \mathfrak{L}_1(k, n, j, l) \cap \mathfrak{P}_{G \smallsetminus G'} \tag{5.5.4}$$

$$\cup \bigcup_{k \geq 0} \bigcup_{n \geq k} \bigcup_{0 \leq j \leq 2n+3} \bigcup_{0 \leq l \leq Z(n+1)} \mathfrak{L}_3(k, n, j, l) \cap \mathfrak{P}_{G \smallsetminus G'} . \tag{5.5.5}$$

PROOF. Let $\mathfrak{p} \in \mathfrak{P}_2 \cap \mathfrak{P}_{G \smallsetminus G'}$. Clearly, for every cube $J = \mathcal{J}(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{P}_{G \smallsetminus G'}$ there exists some $k \geq 0$ such that (5.1.1) holds, and for no cube $J \in \mathcal{D}$ and no $k < 0$ does (5.1.2) hold. Thus $\mathfrak{p} \in \mathfrak{P}(k)$ for some $k \geq 0$.

Next, since $E_2(\lambda, \mathfrak{p}') \subset \mathcal{J}(\mathfrak{p}') \cap G$ for every $\lambda \geq 2$ and every tile $\mathfrak{p}' \in \mathfrak{P}(k)$ with $\lambda \mathfrak{p} \lesssim \lambda \mathfrak{p}'$, it follows from (5.1.2) that $\mu(E_2(\lambda, \mathfrak{p}')) \leq 2^{-k} \mu(\mathcal{J}(\mathfrak{p}'))$ for every such $\mathfrak{p}'$, so $\operatorname{dens}'_k(\{\mathfrak{p}\}) \leq 2^{-k}$. Combining this with $a \geq 0$, it follows from (5.1.7) that there exists $n \geq k$ with $\mathfrak{p} \in \mathfrak{C}(k, n)$.

Since $\mathfrak{p} \in \mathfrak{P}_{G \smallsetminus G'}$, we have in particular $\mathcal{J}(\mathfrak{p}) \not\subset A(2n + 6, k, n)$, so there exist at most $1 + (4n + 12)2^n < 2^{2n+4}$ tiles $\mathfrak{m} \in \mathfrak{M}(k, n)$ with $\mathfrak{p} \leq \mathfrak{m}$. It follows that $\mathfrak{p} \in \mathfrak{L}_0(k, n)$ or $\mathfrak{p} \in \mathfrak{C}_1(k, n, j)$ for some $1 \leq j \leq 2n + 3$. In the former case we are done, in the latter case the inclusion to be shown follows immediately from the definitions of the collections $\mathfrak{C}_i$ and $\mathfrak{L}_i$. $\square$

LEMMA 5.5.2 (L0 antichain). *We have that*

$$\mathfrak{L}_0(k, n) = \bigcup_{1 \leq l \leq n}^{\cdot} \mathfrak{L}_0(k, n, l) ,$$

where each $\mathfrak{L}_0(k, n, l)$ is an antichain.

PROOF. It suffices to show that $\mathfrak{L}_0(k, n)$ contains no chain of length $n + 1$. Suppose that we had such a chain $\mathfrak{p}_0 \leq \mathfrak{p}_1 \leq \cdots \leq \mathfrak{p}_n$ with $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $i = 0, \ldots, n-1$. By (5.1.7), we have that $\operatorname{dens}'_k(\{\mathfrak{p}_n\}) > 2^{-n}$. Thus, by (5.1.6), there exists $\mathfrak{p}' \in \mathfrak{P}(k)$ and $\lambda \geq 2$ with $\lambda \mathfrak{p}_n \leq \lambda \mathfrak{p}'$ and

$$\frac{\mu(E_2(\lambda, \mathfrak{p}'))}{\mu(\mathcal{J}(\mathfrak{p}'))} > \lambda^a 2^{4a} 2^{-n} . \tag{5.5.6}$$

Let $\mathfrak{O}$ be the set of all $\mathfrak{p}'' \in \mathfrak{P}(k)$ such that we have $\mathcal{J}(\mathfrak{p}'') = \mathcal{J}(\mathfrak{p}')$ and $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \lambda) \cap \Omega(\mathfrak{p}'') \neq \varnothing$. We now show that

$$|\mathfrak{O}| \leq 2^{4a} \lambda^a . \tag{5.5.7}$$

The balls $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}''), 0.2)$, $\mathfrak{p}'' \in \mathfrak{O}$ are disjoint by (2.0.15), and by the triangle inequality contained in $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \lambda + 1.2)$. By assumption (1.0.11) on Θ, this ball can be covered with

$$2^{a \lceil \log_2(\lambda + 1.2) + \log_2(5) \rceil} \leq 2^{a(\log_2(\lambda) + 4)} = 2^{4a} \lambda^a$$

many $d_{\mathfrak{p}'}$-balls of radius 0.2. Here we have used that for $\lambda \geq 2$

$$\lceil \log_2(\lambda + 1.2) + \log_2(5) \rceil \leq 1 + \log_2(1.6\lambda) + \log_2(5) = 4 + \log_2(\lambda).$$

By the triangle inequality, each such ball contains at most one $\mathcal{Q}(\mathfrak{p}'')$, and each $\mathcal{Q}(\mathfrak{p}'')$ is contained in one of the balls. Thus we get (5.5.7).

By (2.0.26) and (2.0.27) we have $E_2(\lambda, \mathfrak{p}') \subset \bigcup_{\mathfrak{p}'' \in \mathfrak{O}} E_1(\mathfrak{p}'')$, thus

$$2^{4a}\lambda^a 2^{-n} < \sum_{\mathfrak{p}'' \in \mathfrak{O}} \frac{\mu(E_1(\mathfrak{p}''))}{\mu(\mathcal{J}(\mathfrak{p}''))}.$$

Hence there exists a tile $\mathfrak{p}'' \in \mathfrak{O}$ with

$$\mu(E_1(\mathfrak{p}'')) \geq 2^{-n}\mu(\mathcal{J}(\mathfrak{p}')).$$

By the definition (5.1.5) of $\mathfrak{M}(k, n)$, there exists a tile $\mathfrak{m} \in \mathfrak{M}(k, n)$ with $\mathfrak{p}' \leq \mathfrak{m}$. From (5.5.6), the inclusion $E_2(\lambda, \mathfrak{p}') \subset \mathcal{J}(\mathfrak{p}')$ and $a \geq 1$ we obtain

$$2^n \geq 2^{4a}\lambda^a \geq \lambda.$$

From the triangle inequality, Lemma 2.1.2 and $a \geq 1$, we now obtain for all $\vartheta \in B_{\mathfrak{m}}(\mathcal{Q}(\mathfrak{m}), 1)$ that

$$\begin{aligned}
&d_{\mathfrak{p}_0}(\mathcal{Q}(\mathfrak{p}_0), \vartheta) \\
&\leq d_{\mathfrak{p}_0}(\mathcal{Q}(\mathfrak{p}_0), \mathcal{Q}(\mathfrak{p}_n)) + d_{\mathfrak{p}_0}(\mathcal{Q}(\mathfrak{p}_n), \mathcal{Q}(\mathfrak{p}')) + d_{\mathfrak{p}_0}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p}'')) \\
&\quad + d_{\mathfrak{p}_0}(\mathcal{Q}(\mathfrak{p}''), \mathcal{Q}(\mathfrak{m})) + d_{\mathfrak{p}_0}(\mathcal{Q}(\mathfrak{m}), \vartheta) \\
&\leq 1 + 2^{-95an}(d_{\mathfrak{p}_n}(\mathcal{Q}(\mathfrak{p}_n), \mathcal{Q}(\mathfrak{p}')) + d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p}'')) \\
&\quad + d_{\mathfrak{p}''}(\mathcal{Q}(\mathfrak{p}''), \mathcal{Q}(\mathfrak{m})) + d_{\mathfrak{m}}(\mathcal{Q}(\mathfrak{m}), \vartheta)) \\
&\leq 1 + 2^{-95an}(\lambda + (\lambda + 1) + 1 + 1) \leq 100.
\end{aligned}$$

Thus, by (2.0.23), $100\mathfrak{p}_0 \lesssim \mathfrak{m}$, a contradiction to $\mathfrak{p}_0 \notin \mathfrak{C}(k, n)$. $\qquad\square$

LEMMA 5.5.3 (L2 antichain). *Each of the sets $\mathfrak{L}_2(k, n, j)$ is an antichain.*

PROOF. Suppose that there are $\mathfrak{p}_0, \mathfrak{p}_1 \in \mathfrak{L}_2(k, n, j)$ with $\mathfrak{p}_0 \neq \mathfrak{p}_1$ and $\mathfrak{p}_0 \leq \mathfrak{p}_1$. By Lemma 5.3.1 and Lemma 5.3.2, it follows that $2\mathfrak{p}_0 \lesssim 200\mathfrak{p}_1$. Since $\mathfrak{L}_2(k, n, j)$ is finite, there exists a maximal $l \geq 1$ such that there exists a chain $2\mathfrak{p}_0 \lesssim 200\mathfrak{p}_1 \lesssim \cdots \lesssim 200\mathfrak{p}_l$ with $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $i = 0, \dots, l-1$. If we have $\mathfrak{p}_l \in \mathfrak{U}_1(k, n, j)$, then it follows from $2\mathfrak{p}_0 \lesssim 200\mathfrak{p}_l \lesssim \mathfrak{p}_l$ and (5.1.15) that $\mathfrak{p}_0 \notin \mathfrak{L}_2(k, n, j)$, a contradiction. Thus, by the definition (5.1.14) of $\mathfrak{U}_1(k, n, j)$, there exists $\mathfrak{p}_{l+1} \in \mathfrak{C}_1(k, n, j)$ with $\mathcal{J}(\mathfrak{p}_l) \subsetneq \mathcal{J}(\mathfrak{p}_{l+1})$ and $\vartheta \in B_{\mathfrak{p}_l}(\mathcal{Q}(\mathfrak{p}_l), 100) \cap B_{\mathfrak{p}_{l+1}}(\mathcal{Q}(\mathfrak{p}_{l+1}), 100)$. Using the triangle inequality and Lemma 2.1.2, one deduces that $200\mathfrak{p}_l \lesssim 200\mathfrak{p}_{l+1}$. This contradicts maximality of l. $\qquad\square$

LEMMA 5.5.4 (L1 L3 antichain). *Each of the sets $\mathfrak{L}_1(k, n, j, l)$ and $\mathfrak{L}_3(k, n, j, l)$ is an antichain.*

PROOF. By its definition (5.1.11), each set $\mathfrak{L}_1(k, n, j, l)$ is a set of minimal elements in some set of tiles with respect to $\leq$. If there were distinct $\mathfrak{p}, \mathfrak{q} \in \mathfrak{L}_1(k, n, j, l)$ with $\mathfrak{p} \leq \mathfrak{q}$, then $\mathfrak{q}$ would not be minimal. Hence such $\mathfrak{p}, \mathfrak{q}$ do not exist. Similarly, by (5.1.17), each set $\mathfrak{L}_3(k, n, j, l)$ is a set of maximal elements in some set of tiles

with respect to $\leq$. If there were distinct $\mathfrak{p}, \mathfrak{q} \in \mathfrak{L}_3(k, n, j, l)$ with $\mathfrak{p} \leq \mathfrak{q}$, then $\mathfrak{p}$ would not be maximal. $\qquad \square$

We now turn to the proof of Lemma 5.1.3.

PROOF OF LEMMA 5.1.3. If $\mathfrak{p} \notin \mathfrak{P}_{G \smallsetminus G'}$, then $\mu(\mathcal{J}(\mathfrak{p}) \cap (G \smallsetminus G')) = 0$. By (2.0.21) and (2.0.26), it follows that $\mathbf{1}_{G \smallsetminus G'} T_{\mathfrak{p}} f(x) = 0$. We thus have

$$\mathbf{1}_{G \smallsetminus G'} \sum_{\mathfrak{p} \in \mathfrak{P}_2} T_{\mathfrak{p}} f(x) = \mathbf{1}_{G \smallsetminus G'} \sum_{\mathfrak{p} \in \mathfrak{P}_2 \cap \mathfrak{P}_{G \smallsetminus G'}} T_{\mathfrak{p}} f(x) \,.$$

Let $\mathfrak{L}(k, n)$ denote any of the terms $\mathfrak{L}_i(k, n, j, l) \cap \mathfrak{P}_{G \smallsetminus G'}$ on the right hand side of (5.5.1), where the indices j, l may be void. Then $\mathfrak{L}(k, n)$ is an antichain, by Lemmas 5.5.2, 5.5.3, 5.5.4. Further, we have

$$\mathrm{dens}_1(\mathfrak{L}(k, n)) \leq 2^{4a+1-n}$$

by Lemma 5.3.12, and we have

$$\mathrm{dens}_2(\mathfrak{L}(k, n)) \leq 2^{2a+5} \frac{\mu(F)}{\mu(G)},$$

since

$$\mathfrak{L}(k, n) \cap \mathfrak{P}_{F,G} \subset \mathfrak{P}_{G \smallsetminus G'} \cap \mathfrak{P}_{F,G} = \varnothing.$$

Applying now the triangle inequality according to the decomposition in Lemma 5.5.1, and then applying Proposition 2.0.3 to each term, we obtain the estimate

$$\leq \sum_{k \geq 0} \sum_{n \geq k} (n + (2n + 4) + 2(2n + 4) Z(n + 1))$$

$$\times 2^{201 a^3} (q-1)^{-1} (2^{4a+1-n})^{\frac{q-1}{8a^4}} (2^{2a+5} \frac{\mu(F)}{\mu(G)})^{\frac{1}{q} - \frac{1}{2}} \|f\|_2 \|\mathbf{1}_{G \smallsetminus G'}\|_2 \,.$$

Because $|f| \leq \mathbf{1}_F$, we have $\|f\|_2 \leq \mu(F)^{1/2}$, and we have $\|\mathbf{1}_{G \smallsetminus G'}\|_2 \leq \mu(G)^{1/2}$. Using this and (2.0.3), we bound

$$\leq 2^{202 a^3} (q-1)^{-1} \mu(F)^{\frac{1}{q}} \mu(G)^{\frac{1}{q'}} \sum_{k \geq 0} \sum_{n \geq k} n^2 2^{-n \frac{q-1}{8a^4}} \,.$$

The last sum equals, by changing the order of summation,

$$\sum_{n \geq 0} n^2 (n+1) 2^{-n \frac{q-1}{8a^4}} \leq \frac{2^{8a^3}}{(q-1)^4} \,.$$

This completes the proof. $\qquad \square$

Proof of the Antichain Operator Proposition

Let an antichain $\mathfrak{A}$ and functions f, g as in Proposition 2.0.3 be given. We prove (2.0.30) in Section 6.1 as the geometric mean of two inequalities, each involving one of the two densities. One of these two inequalities will need a careful estimate formulated in Lemma 6.1.5 of the TT^* correlation between two tile operators. Lemma 6.1.5 will be proven in Section 6.2.

The summation of the contributions of these individual correlations will require a geometric Lemma 6.1.6 counting the relevant tile pairs. Lemma 6.1.6 will be proven in Subsection 6.3.

6.1. The density arguments

We begin with the following crucial disjointedness property of the sets $E(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{A}$.

LEMMA 6.1.1 (tile disjointness). *Let $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{A}$. If there exists an $x \in X$ with $x \in E(\mathfrak{p}) \cap E(\mathfrak{p}')$, then $\mathfrak{p} = \mathfrak{p}'$.*

PROOF. Let $\mathfrak{p}, \mathfrak{p}'$ and x be given. Assume without loss of generality that $s(\mathfrak{p}) \leq s(\mathfrak{p}')$. As we have $x \in E(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{p})$ and $x \in E(\mathfrak{p}') \subset \mathcal{J}(\mathfrak{p}')$ by Definition (2.0.20), we conclude for $i = 1, 2$ that $Q(x) \in \Omega(\mathfrak{p})$ and $Q(x) \in \Omega(\mathfrak{p}')$. By (2.0.14) we have $\Omega(\mathfrak{p}') \subset \Omega(\mathfrak{p})$. By Definition (2.0.23), we conclude $\mathfrak{p} \leq \mathfrak{p}'$. As $\mathfrak{A}$ is an antichain, we conclude $\mathfrak{p} = \mathfrak{p}'$. This proves the lemma. $\square$

Let $\mathcal{B}$ be the collection of balls

$$B(c(\mathfrak{p}), 8D^{s(\mathfrak{p})}) \tag{6.1.1}$$

with $\mathfrak{p} \in \mathfrak{A}$ and recall the definition of $M_{\mathcal{B}}$ from Definition 2.0.41.

LEMMA 6.1.2 (maximal bound antichain). *Let $x \in X$. Then*

$$|\sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x)| \leq 2^{107a^3} M_{\mathcal{B}} f(x) . \tag{6.1.2}$$

PROOF. Fix $x \in X$. By Lemma 6.1.1, there is at most one $\mathfrak{p} \in \mathfrak{A}$ such that $T_{\mathfrak{p}} f(x)$ is not zero. If there is no such $\mathfrak{p}$, the estimate (6.1.2) follows.

Assume there is such a $\mathfrak{p}$. By definition of $T_{\mathfrak{p}}$ we have $x \in E(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{p})$ and by the squeezing property (2.0.10)

$$\rho(x, c(\mathfrak{p})) \leq 4D^{s(\mathfrak{p})} . \tag{6.1.3}$$

Let $y \in X$ with $K_{s(\mathfrak{p})}(x, y) \neq 0$. By Definition (2.0.5) of $K_{s(\mathfrak{p})}$ we have

$$\frac{1}{4} D^{s(\mathfrak{p})-1} \leq \rho(x, y) \leq \frac{1}{2} D^{s(\mathfrak{p})} . \qquad (6.1.4)$$

The triangle inequality with (6.1.3) and (6.1.4) implies

$$\rho(c(\mathfrak{p}), y) \leq 8 D^{s(\mathfrak{p})} . \qquad (6.1.5)$$

Using the kernel bound (1.0.14) and the lower bound in (2.1.2) we obtain

$$|K_{s(\mathfrak{p})}(x, y)| \leq \frac{2^{a^3}}{\mu(B(x, \frac{1}{4} D^{s(\mathfrak{p})-1}))} . \qquad (6.1.6)$$

Using $D = 2^{100a^2}$ and the doubling property (1.0.5) $5 + 100a^2$ times estimates the last display by

$$\leq \frac{2^{5a+101a^3}}{\mu(B(x, 8D^{s(\mathfrak{p})}))} . \qquad (6.1.7)$$

Using that $|e(\vartheta)|$ is bounded by 1 for every $\vartheta \in \Theta$, we estimate with the triangle inequality and the above information

$$|T_{\mathfrak{p}} f(x)| \leq \frac{2^{5a+101a^3}}{\mu(B(x, 8D^{s(\mathfrak{p})}))} \int_{\mu(B(x, 8D^{s(\mathfrak{p})}))} |f(y)| \, dy \qquad (6.1.8)$$

This together with $a \geq 1$ proves the Lemma. $\qquad \square$

Set

$$\tilde{q} = \frac{2q}{1+q} . \qquad (6.1.9)$$

Since $1 < q \leq 2$, we have $1 < \tilde{q} < q \leq 2$.

Lemma 6.1.3 (dens2 antichain). *We have that*

$$\left| \int \overline{g(x)} \sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x) \, d\mu(x) \right| \leq 2^{111a^2} (q-1)^{-1} \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{q}-\frac{1}{2}} \|f\|_2 \|g\|_2 . \qquad (6.1.10)$$

Proof. We have $f = \mathbf{1}_F f$. Using Hölder's inequality, we obtain for each $x \in B'$ and each $B' \in \mathcal{B}$ using $1 < \tilde{q} \leq 2$

$$\frac{1}{\mu(B')} \int_{B'} |f(y)| \, d\mu(y) \qquad (6.1.11)$$

$$\leq \left(\frac{1}{\mu(B')} \int_{B'} |f(y)|^{\frac{2\tilde{q}}{3\tilde{q}-2}} \, d\mu(y) \right)^{\frac{3}{2}-\frac{1}{\tilde{q}}} \left(\frac{1}{\mu(B')} \int_{B'} \mathbf{1}_F(y) \, d\mu(y) \right)^{\frac{1}{\tilde{q}}-\frac{1}{2}} \qquad (6.1.12)$$

$$\leq \left(M_{\mathcal{B}}(|f|^{\frac{2\tilde{q}}{3\tilde{q}-2}})(x) \right)^{\frac{3}{2}-\frac{1}{\tilde{q}}} \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{\tilde{q}}-\frac{1}{2}} . \qquad (6.1.13)$$

Taking the maximum over all B' containing x, we obtain

$$M_{\mathcal{B}} |f| \leq M_{\mathcal{B}, \frac{2\tilde{q}}{3\tilde{q}-2}} |f| \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{\tilde{q}}-\frac{1}{2}} . \qquad (6.1.14)$$

We have with Proposition 2.0.6

$$\left\| M_{\mathcal{B}, \frac{2q}{3q-2}} f \right\|_2 \leq 2^{2a} (3\tilde{q} - 2)(2\tilde{q} - 2)^{-1} \|f\|_2 . \qquad (6.1.15)$$

Using $1 < \tilde{q} \le 2$ estimates the last display by

$$2^{2a+2}(\tilde{q}-1)^{-1}\|f\|_2\,. \tag{6.1.16}$$

We obtain with Cauchy-Schwarz and then Lemma 6.1.2

$$|\int \overline{g(x)} \sum_{\mathfrak{p}\in\mathfrak{A}} T_\mathfrak{p} f(x)\,d\mu(x)| \tag{6.1.17}$$

$$\le \|g\|_2 \Big\|\sum_{\mathfrak{p}\in\mathfrak{A}} T_\mathfrak{p} f\Big\|_2 \tag{6.1.18}$$

$$\le 2^{107a^2}\|g\|_2\|M_\mathcal{B} f\|_2 \tag{6.1.19}$$

With (6.1.14) and (6.1.16) we can estimate the last display by

$$\le 2^{107a^2+2a+2}(\tilde{q}-1)^{-1}\|g\|_2\|f\|_2\,\mathrm{dens}_2(\mathfrak{A})^{\frac{1}{\tilde{q}}-\frac{1}{2}} \tag{6.1.20}$$

Using $a \ge 4$ and $(\tilde{q}-1)^{-1} = (q+1)/(q-1) \le 3(q-1)^{-1}$ proves the lemma. $\qquad\square$

LEMMA 6.1.4 (dens1 antichain). *Set $p := 4a^4$. We have*

$$\left|\int \overline{g(x)} \sum_{\mathfrak{p}\in\mathfrak{A}} T_\mathfrak{p} f(x)\,d\mu(x)\right| \le 2^{150a^3}\,\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{2p}}\|f\|_2\|g\|_2\,. \tag{6.1.21}$$

PROOF. We write for the expression inside the absolute values on the left-hand side of (6.1.21)

$$\sum_{\mathfrak{p}\in\mathfrak{A}} \iint \overline{g(x)}\mathbf{1}_{E(\mathfrak{p})}(x)K_{s(\mathfrak{p})}(x,y)e(Q(x)(y) - Q(x)(x))f(y)\,d\mu(y)\,d\mu(x) \tag{6.1.22}$$

$$= \int \sum_{\mathfrak{p}\in\mathfrak{A}} \overline{T_\mathfrak{p}^* g(y)}f(y)\,d\mu(y) \tag{6.1.23}$$

with the adjoint operator

$$T_\mathfrak{p}^* g(y) = \int_{E(\mathfrak{p})} \overline{K_{s(\mathfrak{p})}(x,y)}e(-Q(x)(y) + Q(x)(x))g(x)\,d\mu(x)\,. \tag{6.1.24}$$

We have by expanding the square

$$\int \Big|\sum_{\mathfrak{p}\in\mathfrak{A}} T_\mathfrak{p}^* g(y)\Big|^2 d\mu(y) = \int \Big(\sum_{\mathfrak{p}\in\mathfrak{A}} T_\mathfrak{p}^* g(y)\Big)\Big(\sum_{\mathfrak{p}'\in\mathfrak{A}} \overline{T_{\mathfrak{p}'}^* g(y)}\Big)\,d\mu(y) \tag{6.1.25}$$

$$\le \sum_{\mathfrak{p}\in\mathfrak{A}}\sum_{\mathfrak{p}'\in\mathfrak{A}} \Big|\int T_\mathfrak{p}^* g(y)\overline{T_{\mathfrak{p}'}^* g(y)}\,d\mu(y)\Big|\,. \tag{6.1.26}$$

We split the sum into the terms with $s(\mathfrak{p}') \le s(\mathfrak{p})$ and $s(\mathfrak{p}) < s(\mathfrak{p}')$. Using the symmetry of each summand, we may switch $\mathfrak{p}$ and $\mathfrak{p}'$ in the second sum. Using further positivity of each summand to replace the condition $s(\mathfrak{p}') < s(\mathfrak{p})$ by $s(\mathfrak{p}') \le s(\mathfrak{p})$ in the second sum, we estimate (6.1.26) by

$$\le 2\sum_{\mathfrak{p}\in\mathfrak{A}}\sum_{\mathfrak{p}'\in\mathfrak{A}:s(\mathfrak{p}')\le s(\mathfrak{p})} \Big|\int T_\mathfrak{p}^* g(y)\overline{T_{\mathfrak{p}'}^* g(y)}\,d\mu(y)\Big|\,. \tag{6.1.27}$$

Define for $\mathfrak{p} \in \mathfrak{P}$

$$B(\mathfrak{p}) := B(\mathrm{c}(\mathfrak{p}), 15D^{\mathrm{s}(\mathfrak{p})}) \tag{6.1.28}$$

and define

$$\mathfrak{A}(\mathfrak{p}) := \{\mathfrak{p}' \in \mathfrak{A} : \mathrm{s}(\mathfrak{p}') \le \mathrm{s}(\mathfrak{p}) \wedge \mathcal{I}(\mathfrak{p}') \subset B(\mathfrak{p})\}. \tag{6.1.29}$$

Note that by the squeezing property (2.0.10) and the doubling property (1.0.5) applied 6 times we have

$$\mu(B(\mathfrak{p})) \le 2^{6a}\mu(\mathcal{I}(\mathfrak{p})). \tag{6.1.30}$$

Using Lemma 6.1.5 and (6.1.30), we estimate (6.1.27) by

$$\le 2^{255a^3+6a+1} \sum_{\mathfrak{p} \in \mathfrak{A}} \int_{E(\mathfrak{p})} |g|(y)h(\mathfrak{p}) \, d\mu(y) \tag{6.1.31}$$

with $h(\mathfrak{p})$ defined as

$$\frac{1}{\mu(B(\mathfrak{p}))} \int \sum_{\mathfrak{p}' \in \mathfrak{A}(\mathfrak{p})} (1 + d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p}))^{-1/(2a^2+a^3)}(\mathbf{1}_{E(\mathfrak{p}')}|g|)(y') \, d\mu(y'). \tag{6.1.32}$$

Note that $p \ge 4$ since $a > 4$. We estimate $h(\mathfrak{p})$ as defined in (6.1.32) with Hölder using $|g| \le \mathbf{1}_G$ and $E(\mathfrak{p}') \subset B(\mathfrak{p})$ by

$$\frac{\|g\mathbf{1}_{B(\mathfrak{p})}\|_{p'}}{\mu(B(\mathfrak{p}))} \left\| \sum_{\mathfrak{p} \in \mathfrak{A}(\mathfrak{p})} (1 + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}'))^{-1/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \right\|_p. \tag{6.1.33}$$

Then we apply Lemma 6.1.6 to estimate this by

$$\le 2^{104a} \frac{\|g\mathbf{1}_{B(\mathfrak{p})}\|_{p'}}{\mu(B(\mathfrak{p}))} \mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}} \mu(B(\mathfrak{p}))^{\frac{1}{p}}. \tag{6.1.34}$$

Let $\mathcal{B}'$ be the collection of all balls $B(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{A}$. Then for each $\mathfrak{p} \in \mathfrak{A}$ and $x \in B(\mathfrak{p})$ we have by definition (2.0.41) of $M_{\mathcal{B}',p'}$

$$\|g\mathbf{1}_{B(\mathfrak{p})}\|_{p'} \le \mu(B(\mathfrak{p}))^{\frac{1}{p'}} M_{\mathcal{B}',p'}g(x). \tag{6.1.35}$$

Hence we can estimate (6.1.34) by

$$\le 2^{104a}(M_{\mathcal{B}',p'}g(x)) \mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}}. \tag{6.1.36}$$

With this estimate of $h(\mathfrak{p})$, using $E(\mathfrak{p}) \subset B(\mathfrak{p})$ by construction of $B(\mathfrak{p})$, we estimate (6.1.31) by

$$\le 2^{255a^3+110a+1}\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}} \sum_{\mathfrak{p} \in \mathfrak{A}} \int_{E(\mathfrak{p})} |g|(y)M_{\mathcal{B}',p'}g(y) \, dy. \tag{6.1.37}$$

Using Lemma 6.1.1, the last display is observed to be

$$= 2^{255a^3+110a+1}\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}} \int |g|(y)(M_{\mathcal{B}',p'}g)(y) \, dy. \tag{6.1.38}$$

Applying Cauchy-Schwarz and using Proposition 2.0.6 estimates the last display by

$$2^{255a^3+110a+1} \mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}} \|g\|_2 \|M_{\mathcal{B}',p'}g\|_2 \tag{6.1.39}$$

$$\le 2^{255a^3+110a+3} \frac{2}{2-p'} \mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}} \|g\|_2^2. \tag{6.1.40}$$

Using $p > 4$ and thus $1 < p' < \frac{4}{3}$, we estimate the last display by

$$\leq 2^{255a^3 + 110a + 5} \operatorname{dens}_1(\mathfrak{A})^{\frac{1}{p}} \|g\|_2^2. \tag{6.1.41}$$

Now Lemma 6.1.4 follows by applying Cauchy-Schwarz on the left-hand side and using $a \geq 4$. $\qquad\square$

The following basic TT^* estimate will be proved in Section 6.2.

LEMMA 6.1.5 (tile correlation). *Let* $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}$ *with* $\mathrm{s}(\mathfrak{p}') \leq \mathrm{s}(\mathfrak{p})$. *Then*

$$\left| \int T_{\mathfrak{p}'}^* g \, \overline{T_{\mathfrak{p}}^* g} \right| \tag{6.1.42}$$

$$\leq 2^{255a^3} \frac{(1 + d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p}))^{-1/(2a^2 + a^3)}}{\mu(\mathcal{J}(\mathfrak{p}))} \int_{E(\mathfrak{p}')} |g| \int_{E(\mathfrak{p})} |g|. \tag{6.1.43}$$

Moreover, the term (6.1.42) *vanishes unless*

$$\mathcal{J}(\mathfrak{p}') \subset B(\mathrm{c}(\mathfrak{p}), 15 D^{\mathrm{s}(\mathfrak{p})}). \tag{6.1.44}$$

The following lemma will be proved in Section 6.3.

LEMMA 6.1.6 (antichain tile count). *Set* $p := 4a^4$ *and let* p' *be the dual exponent of* p, *that is* $1/p + 1/p' = 1$. *For every* $\vartheta \in \Theta$ *and every subset* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *we have*

$$\left\| \sum_{\mathfrak{p} \in \mathfrak{A}'} (1 + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta))^{-1/(2a^2 + a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \right\|_p \tag{6.1.45}$$

$$\leq 2^{104a} \operatorname{dens}_1(\mathfrak{A})^{\frac{1}{p}} \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}'} I_{\mathfrak{p}} \right)^{\frac{1}{p}}. \tag{6.1.46}$$

From these lemmas it is easy to prove Proposition 2.0.3.

PROOF OF PROPOSITION 2.0.3. We have

$$\left(\frac{1}{\tilde{q}} - \frac{1}{2} \right) (2 - q) = \frac{1}{q} - \frac{1}{2}. \tag{6.1.47}$$

Multiplying the $(2-q)$-th power of (6.1.10) and the $(q-1)$-th power of (6.1.21) and estimating gives after simplification of some factors

$$\left| \int \overline{g(x)} \sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x) \, d\mu(x) \right| \tag{6.1.48}$$

$$\leq 2^{150a^3} (q-1)^{-1} \operatorname{dens}_1(\mathfrak{A})^{\frac{q-1}{2p}} \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{q} - \frac{1}{2}} \|f\|_2 \|g\|_2. \tag{6.1.49}$$

With the definition of p, this implies Proposition 2.0.3. $\qquad\square$

6.2. Proof of the Tile Correlation Lemma

The next lemma prepares an application of Proposition 2.0.5.

LEMMA 6.2.1 (correlation kernel bound). *Let* $-S \leq s_1 \leq s_2 \leq S$ *and let* $x_1, x_2 \in X$. *Define*

$$\varphi(y) := \overline{K_{s_1}(x_1, y)} K_{s_2}(x_2, y) . \tag{6.2.1}$$

If $\varphi(y) \neq 0$, *then*

$$y \in B(x_1, D^{s_1}) . \tag{6.2.2}$$

Moreover, we have with $\tau = 1/a$

$$\|\varphi\|_{C^\tau(B(x_1, D^{s_1}))} \leq \frac{2^{254a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))} . \tag{6.2.3}$$

PROOF. If $\varphi(y)$ is not zero, then $K_{s_1}(x_1, y)$ is not zero and thus (2.1.2) gives (6.2.2).

We next have for y with (2.1.3)

$$|\varphi(y)| \leq \frac{2^{204a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))} \tag{6.2.4}$$

and for $y' \neq y$ additionally with (2.1.4)

$$|\varphi(y) - \varphi(y')| \tag{6.2.5}$$

$$\leq |K_{s_1}(x_1, y) - K_{s_1}(x_1, y')||K_{s_2}(x_2, y)| \tag{6.2.6}$$

$$+ |K_{s_1}(x_1, y')||K_{s_2}(x_2, y) - K_{s_2}(x_2, y')| \tag{6.2.7}$$

$$\leq \frac{2^{252a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))} \left(\left(\frac{\rho(y, y')}{D^{s_1}} \right)^{1/a} + \left(\frac{\rho(y, y')}{D^{s_2}} \right)^{1/a} \right) \tag{6.2.8}$$

$$\leq \frac{2^{253a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))} \left(\frac{\rho(y, y')}{D^{s_1}} \right)^{1/a} . \tag{6.2.9}$$

Adding the estimates (6.2.4) and (6.2.9) gives (6.2.3). This proves the lemma. $\qquad\square$

The next lemma is a geometric estimate for two tiles.

LEMMA 6.2.2. *Let* $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}$ *with* $s(\mathfrak{p}_1) \leq s(\mathfrak{p}_2)$. *For each* $x_1 \in E(\mathfrak{p}_1)$ *and* $x_2 \in E(\mathfrak{p}_2)$ *we have*

$$1 + d_{\mathfrak{p}_1}(\mathcal{Q}(\mathfrak{p}_1), \mathcal{Q}(\mathfrak{p}_2)) \leq 2^{3a}(1 + d_{B(x_1, D^{s(\mathfrak{p}_1)})}(Q(x_1), Q(x_2))) . \tag{6.2.10}$$

PROOF. Let $i \in \{1, 2\}$. By Definition (2.0.20) of E, we have $Q(x_i) \in \Omega(\mathfrak{p}_i)$ With (2.0.15) we then conclude

$$d_{\mathfrak{p}_i}(Q(x_i), \mathcal{Q}(\mathfrak{p}_i)) \leq 1 . \tag{6.2.11}$$

We have $\mathcal{I}(\mathfrak{p}_1) \subset \mathcal{I}(\mathfrak{p}_2)$ by (2.0.8). Using Lemma 2.1.2 it follows that

$$d_{\mathfrak{p}_1}(Q(x_2), \mathcal{Q}(\mathfrak{p}_2)) \leq 1 . \tag{6.2.12}$$

By the triangle inequality, we obtain from (6.2.11) and (6.2.12)

$$1 + d_{\mathfrak{p}_1}(\mathcal{Q}(\mathfrak{p}_1), \mathcal{Q}(\mathfrak{p}_2)) \leq 3 + d_{\mathfrak{p}_1}(Q(x_1), Q(x_2)) . \tag{6.2.13}$$

As $x_1 \in \mathcal{J}(\mathfrak{p}_1)$ by Definition (2.0.20) of E, we have by the squeezing property (2.0.10)

$$d(x_1, \mathrm{c}(\mathfrak{p}_1)) \leq 4D^{\mathrm{s}(\mathfrak{p}_1)} \tag{6.2.14}$$

and thus by (2.0.10) again and the triangle inequality

$$\mathcal{J}(\mathfrak{p}_1) \subset B(x_1, 8D^{\mathrm{s}(\mathfrak{p}_1)}). \tag{6.2.15}$$

We thus estimate the right-hand side of (6.2.13) with monotonicity (1.0.9) of the metrics d_B by

$$\leq 3 + d_{B(x_1, 8D^{\mathrm{s}(\mathfrak{p}_1)})}(Q(x_1), Q(x_2)). \tag{6.2.16}$$

This is further estimated by applying the doubling property (1.0.8) three times by

$$\leq 3 + 2^{3a} d_{B_1(x_1, D^{\mathrm{s}(\mathfrak{p}_1)})}(Q(x_1), Q(x_2)). \tag{6.2.17}$$

Now (6.2.10) follows with $a \geq 1$. $\qquad\square$

LEMMA 6.2.3 (tile range support). *For each $\mathfrak{p} \in \mathfrak{P}$, and each $y \in X$, we have that*

$$T_{\mathfrak{p}}^* g(y) \neq 0 \tag{6.2.18}$$

implies

$$y \in B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})}). \tag{6.2.19}$$

PROOF. Fix $\mathfrak{p}$ and y with (6.2.18). Then there exists $x \in E(\mathfrak{p})$ with

$$\overline{K_{\mathrm{s}(\mathfrak{p})}(x, y)} e(-Q(x)(y) + Q(x)(x)) g(x) \neq 0. \tag{6.2.20}$$

As $E(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{p})$ and by the squeezing property (2.0.10), we have

$$\rho(x, \mathrm{c}(\mathfrak{p})) \leq 4D^{\mathrm{s}(\mathfrak{p})}. \tag{6.2.21}$$

As $K_{\mathrm{s}(\mathfrak{p})}(x, y) \neq 0$, we have by (2.1.2) that

$$\rho(x, y) \leq \frac{1}{2} D^{\mathrm{s}(\mathfrak{p})}. \tag{6.2.22}$$

Now (6.2.19) follows by the triangle inequality. $\qquad\square$

We now prove Lemma 6.1.5. We begin with (6.1.42).

We expand the left-hand side of (6.1.42) as

$$\left| \int \int_{E(\mathfrak{p}_1)} \overline{K_{\mathrm{s}(\mathfrak{p}_1)}(x_1, y)} e(-Q(x_1)(y) + Q(x_1)(x_1)) g(x_1) \, d\mu(x_1) \right. \tag{6.2.23}$$

$$\left. \times \int_{E(\mathfrak{p}_2)} K_{\mathrm{s}(\mathfrak{p}_2)}(x_2, y) e(Q(x_2)(y) - Q(x_2)(x_2)) \overline{g(x_2)} \, d\mu(x_2) \, d\mu(y) \right|. \tag{6.2.24}$$

By Fubini and the triangle inequality and the fact $|e(Q(x_i)(x_i))| = 1$ for $i = 1, 2$, we can estimate (6.2.23) from above by

$$\int_{E(\mathfrak{p}_1)} \int_{E(\mathfrak{p}_2)} \mathbf{I}(x_1, x_2) \, d\mu(x_1) d\mu(x_2). \tag{6.2.25}$$

with

$$\mathbf{I}(x_1, x_2) := \left| \int e(-Q(x_1)(y) + Q(x_2)(y)) \varphi_{x_1, x_2}(y) d\mu(y) \, g(x_1) g(x_2) \right| \tag{6.2.26}$$

We estimate for fixed $x_1 \in E(\mathfrak{p}_1)$ and $x_2 \in E(\mathfrak{p}_2)$ the inner integral of (6.2.25) with Proposition 2.0.5. The function $\varphi := \varphi_{x_1,x_2}$ satisfies the assumptions of Proposition 2.0.5 with $z = x_1$ and $R = D^{s_1}$ by Lemma 6.2.1. We obtain with $B' := B(x_1, D^{s(\mathfrak{p}_1)})$,

$$\mathbf{I}(x_1, x_2) \leq 2^{8a} \mu(B') \|\varphi\|_{C^\tau(B')} (1 + d_{B'}(Q(x_1), Q(x_2)))^{-1/(2a^2+a^3)}$$

$$\leq \frac{2^{254a^3+8a}}{\mu(B(x_2, D^{s(\mathfrak{p}_2)}))} (1 + d_{B'}(Q(x_1), Q(x_2)))^{-1/(2a^2+a^3)}. \tag{6.2.27}$$

Using Lemma 6.2.2 and $a \geq 1$ estimates (6.2.27) by

$$\leq \frac{2^{254a^3+8a+1}}{\mu(B(x_2, D^{s(\mathfrak{p}_2)}))} (1 + d_{\mathfrak{p}_1}(\mathcal{Q}(\mathfrak{p}_1), \mathcal{Q}(\mathfrak{p}_2)))^{-1/(2a^2+a^3)}. \tag{6.2.28}$$

As $x_2 \in \mathcal{J}(\mathfrak{p}_2)$ by Definition (2.0.20) of E, we have by (2.0.10)

$$\rho(x_2, c(\mathfrak{p}_2)) \leq 4D^{s(\mathfrak{p}_2)} \tag{6.2.29}$$

and thus by (2.0.10) again and the triangle inequality

$$\mathcal{J}(\mathfrak{p}_2) \subset B(x_2, 8D^{s(\mathfrak{p}_2)}). \tag{6.2.30}$$

Using three iterations of the doubling property (1.0.5) give

$$\mu(\mathcal{J}(\mathfrak{p}_2)) \leq 2^{3a} \mu(B(x_2, D^{s(\mathfrak{p}_2)})). \tag{6.2.31}$$

With $a \geq 1$ and (6.2.28) we conclude (6.1.42).

Now assume the left-hand side of (6.1.42) is not zero. There is a $y \in X$ with

$$T_{\mathfrak{p}}^* g(y) \overline{T_{\mathfrak{p}'}^* g(y)} \neq 0 \tag{6.2.32}$$

By the triangle inequality and Lemma 6.2.3, we conclude

$$\rho(c(\mathfrak{p}), c(\mathfrak{p}')) \leq \rho(c(\mathfrak{p}), y) + \rho(c(\mathfrak{p}'), y) \leq 5D^{s(\mathfrak{p})} + 5D^{s(\mathfrak{p}')} \leq 10D^{s(\mathfrak{p})}. \tag{6.2.33}$$

By the squeezing property (2.0.10) and the triangle inequality, we conclude

$$\mathcal{J}(\mathfrak{p}') \subset B(c(\mathfrak{p}), 15D^{s(\mathfrak{p})}). \tag{6.2.34}$$

This completes the proof of Lemma 6.1.5.

6.3. Proof of the Antichain Tile Count Lemma

LEMMA 6.3.1 (tile reach). *Let $\vartheta \in \Theta$ and $N \geq 0$ be an integer. Let $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}$ with*

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta)) \leq 2^N \tag{6.3.1}$$

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta)) \leq 2^N. \tag{6.3.2}$$

Assume $\mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{p}')$ and $s(\mathfrak{p}) < s(\mathfrak{p}')$. Then

$$2^{N+2}\mathfrak{p} \lesssim 2^{N+2}\mathfrak{p}'. \tag{6.3.3}$$

PROOF. By Lemma 2.1.2, we have

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \vartheta) \leq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta) \leq 2^N \,. \tag{6.3.4}$$

Together with (6.3.1) and the triangle inequality, we obtain

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p})) \leq 2^{N+1} \,. \tag{6.3.5}$$

Now assume

$$\vartheta' \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 2^{N+2}). \tag{6.3.6}$$

By the doubling property (1.0.8), applied five times, we have

$$d_{B(\mathrm{c}(\mathfrak{p}'), 8D^{\mathrm{s}(\mathfrak{p}')})}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^{5a+N+2} \,. \tag{6.3.7}$$

We have by the squeezing property (2.0.10)

$$\mathrm{c}(\mathfrak{p}) \in B(\mathrm{c}(\mathfrak{p}'), 4D^{\mathrm{s}(\mathfrak{p}')}) \,. \tag{6.3.8}$$

Hence by the triangle inequality

$$B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p}')}) \subset B(\mathrm{c}(\mathfrak{p}'), 8D^{\mathrm{s}(\mathfrak{p}')}) \,. \tag{6.3.9}$$

Together with (6.3.7) and monotonicity (1.0.9) of d

$$d_{B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p}')})}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^{5a+N+2} \,. \tag{6.3.10}$$

Using the doubling property (1.0.10) $5a + 2$ times gives

$$d_{B(\mathrm{c}(\mathfrak{p}), 2^{2-5a^2-2a}D^{\mathrm{s}(\mathfrak{p}')})}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^N \,. \tag{6.3.11}$$

Using $\mathrm{s}(\mathfrak{p}) < \mathrm{s}(\mathfrak{p}')$ and $D = 2^{100a^2}$ and $a \geq 4$ gives

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^N \,. \tag{6.3.12}$$

With the triangle inequality and (6.3.5),

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta') < 2^{N+2} \,. \tag{6.3.13}$$

This shows

$$B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 2^{N+2}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 2^{N+2}) \,. \tag{6.3.14}$$

This implies (6.3.3) and completes the proof of the lemma. $\qquad\square$

For $\vartheta \in \Theta$ and $N \geq 0$ define

$$\mathfrak{A}_{\vartheta, N} := \{\mathfrak{p} \in \mathfrak{A} : 2^N \leq 1 + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta) \leq 2^{N+1}\} \,. \tag{6.3.15}$$

LEMMA 6.3.2 (stack density). *Let $\vartheta \in \Theta$, $N \geq 0$ and $L \in \mathcal{D}$. Then*

$$\sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta, N} : \mathcal{I}(\mathfrak{p}) = L} \mu(E(\mathfrak{p}) \cap G) \leq 2^{a(N+5)} \operatorname{dens}_1(\mathfrak{A}) \mu(L) \,. \tag{6.3.16}$$

PROOF. Let ϑ, N, L be given and set

$$\mathfrak{A}' := \{\mathfrak{p} \in \mathfrak{A}_{\vartheta, N} : \mathcal{I}(\mathfrak{p}) = L\} \,. \tag{6.3.17}$$

Let $\mathfrak{p} \in \mathfrak{A}'$. We have by Definition (2.0.28) using $\lambda = 2$ and the squeezing property (2.0.15)

$$\mu(E(\mathfrak{p}) \cap G) \leq \mu(E_2(2, \mathfrak{p})) \leq 2^a \operatorname{dens}_1(\mathfrak{A}') \mu(L) \,. \tag{6.3.18}$$

By the covering property (1.0.11), applied $N + 4$ times, there is a collection Θ' of at most $2^{a(N+4)}$ elements such that

$$B_{\mathfrak{p}}(\vartheta, 2^{N+1}) \subset \bigcup_{\vartheta' \in \Theta'} B_{\mathfrak{p}}(\vartheta', 0.2) . \tag{6.3.19}$$

As each $\mathcal{Q}(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}$ is contained in the left-hand-side of (6.3.19) by definition, it is in at least one $B_{\mathfrak{p}}(\vartheta', 0.2)$ with $\vartheta' \in \Theta'$.

For two different $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{A}'$, we have by (2.0.13) that $\Omega(\mathfrak{p})$ and $\Omega(\mathfrak{p}')$ are disjoint and thus by the squeezing property (2.0.15) we have for every $\vartheta' \in \Theta'$

$$\vartheta' \notin B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \cap B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), 0.2) . \tag{6.3.20}$$

Hence at most one of $\mathcal{Q}(\mathfrak{p})$ and $\mathcal{Q}(\mathfrak{p})$ is in $B_{\mathfrak{p}}(\vartheta', 0.2)$. It follows that there are at most $2^{a(N+4)}$ elements in $\mathfrak{A}'$. Adding (6.3.18) over $\mathfrak{A}'$ proves (6.3.16). $\qquad\square$

LEMMA 6.3.3 (local antichain density). *Let $\vartheta \in \Theta$ and N be an integer. Let $\mathfrak{p}_\vartheta$ be a tile with $\vartheta \in \Omega(\mathfrak{p}_\vartheta)$. Then we have*

$$\sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N} : s(\mathfrak{p}_\vartheta) < s(\mathfrak{p})} \mu(E(\mathfrak{p}) \cap G \cap \mathcal{I}(\mathfrak{p}_\vartheta)) \leq \mu(E_2(2^{N+3}, \mathfrak{p}_\vartheta)) . \tag{6.3.21}$$

PROOF. Let $\mathfrak{p}$ be any tile in $\mathfrak{A}_{\vartheta,N}$ with $s(\mathfrak{p}_\vartheta) < s(\mathfrak{p})$. By definition of E, the tile contributes zero to the sum on the left-hand side of (6.3.21) unless $\mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{p}_\vartheta) \neq \varnothing$, which we may assume. With $s(\mathfrak{p}_\vartheta) < s(\mathfrak{p})$ and the dyadic property (2.0.8) we conclude $\mathcal{I}(\mathfrak{p}_\vartheta) \subset \mathcal{I}(\mathfrak{p})$. By the squeezing property (2.0.15), we conclude from $\vartheta \in \Omega(\mathfrak{p}_\vartheta)$ that

$$\vartheta \in B(\mathcal{Q}(\mathfrak{p}_\vartheta), 1) . \tag{6.3.22}$$

We conclude from $\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}$ that

$$\vartheta \in B(\mathcal{Q}(\mathfrak{p}), 2^{N+1}) . \tag{6.3.23}$$

With Lemma 6.3.1, we conclude

$$2^{N+3}\mathfrak{p}_\vartheta \lesssim 2^{N+3}\mathfrak{p} . \tag{6.3.24}$$

By Definition (2.0.27) of E_2, we conclude

$$E(\mathfrak{p}) \cap G \subset E_2(2^{N+3}, \mathfrak{p}_\vartheta) . \tag{6.3.25}$$

Using disjointedness of the various $E(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{A}$ by Lemma 6.1.1, we obtain (6.3.21). This proves the lemma. $\qquad\square$

LEMMA 6.3.4 (global antichain density). *Let $\vartheta \in \Theta$ and let $N \geq 0$ be an integer. Then we have*

$$\sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}} \mu(E(\mathfrak{p}) \cap G) \leq 2^{101a^3 + Na} \operatorname{dens}_1(\mathfrak{A}) \mu\left(\cup_{\mathfrak{p} \in \mathfrak{A}} I_{\mathfrak{p}}\right) . \tag{6.3.26}$$

PROOF. Fix ϑ and N. Let $\mathfrak{A}'$ be the set of $\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}$ such that $\mathcal{I}(\mathfrak{p}) \cap G$ is not empty.

Let $\mathcal{L}$ be the collection of dyadic cubes $I \in \mathcal{D}$ such that $I \subset \mathcal{I}(\mathfrak{p})$ for some $\mathfrak{p} \in \mathfrak{A}'$ and if $\mathcal{I}(\mathfrak{p}) \subset I$ for some $\mathfrak{p} \in \mathfrak{A}'$, then $s(\mathfrak{p}) = -S$. By (2.0.7), for each $\mathfrak{p} \in \mathfrak{A}'$

and each $x \in \mathcal{I}(\mathfrak{p}) \cap G$, there is $I \in \mathcal{D}$ with $s(I) = -S$ and $x \in I$. By (2.0.8), we have $I \subset \mathcal{I}(\mathfrak{p})$. Hence

$$\mathcal{I}(\mathfrak{p}) \subset \bigcup \{I \in \mathcal{D} : s(I) = -S, I \subset \mathcal{I}(\mathfrak{p})\} \subset \bigcup \mathcal{L}. \tag{6.3.27}$$

As each $I \in \mathcal{L}$ satisfies $I \subset \mathcal{I}(\mathfrak{p})$ for some $\mathfrak{p}$ in $\mathfrak{A}'$, we conclude

$$\bigcup \mathcal{L} = \bigcup_{\mathfrak{p} \in \mathfrak{A}'} \mathcal{I}(\mathfrak{p}). \tag{6.3.28}$$

Let $\mathcal{L}^*$ be the set of maximal elements in $\mathcal{L}$ with respect to set inclusion. By (2.0.8), the elements in $\mathcal{L}^*$ are pairwise disjoint and we have

$$\bigcup \mathcal{L}^* = \bigcup_{\mathfrak{p} \in \mathfrak{A}'} \mathcal{I}(\mathfrak{p}). \tag{6.3.29}$$

Using the partition (6.3.29) into elements of $\mathcal{L}$ in (6.3.30), it suffices to show for each $L \in \mathcal{L}^*$

$$\sum_{\mathfrak{p} \in \mathfrak{A}'} \mu(E(\mathfrak{p}) \cap G \cap L) \leq 2^{101a^3 + aN} \operatorname{dens}_1(\mathfrak{A}) \mu(L). \tag{6.3.30}$$

Fix $L \in \mathcal{L}^*$. By definition of L, there exists an element $\mathfrak{p}' \in \mathfrak{A}'$ such that $L \subset \mathcal{I}(\mathfrak{p}')$. Pick such an element $\mathfrak{p}'$ in $\mathfrak{A}$ with minimal $s(\mathfrak{p}')$. As $\mathcal{I}(\mathfrak{p}') \not\subset L$ or $s(L) = -S$ by definition of L, we have with (2.0.8) that $s(L) < s(\mathfrak{p}')$ or $s(L) = -S$. In particular $s(L) < S$, thus $L \neq I_0$ and hence by (2.0.9) there exists a cube $J \in \mathcal{D}$ with $L \subsetneq J$. By (2.0.7), there is an $L' \in \mathcal{D}$ with $s(L') = s(L) + 1$ and $c(L) \in L'$. By (2.0.8), we have $L \subset L'$.

We split the left-hand side of (6.3.30) as

$$\sum_{\mathfrak{p} \in \mathfrak{A}' : \mathcal{I}(\mathfrak{p}) = L'} \mu(E(\mathfrak{p}) \cap G \cap L) \tag{6.3.31}$$

$$+ \sum_{\mathfrak{p} \in \mathfrak{A}' : \mathcal{I}(\mathfrak{p}) \neq L'} \mu(E(\mathfrak{p}) \cap G \cap L), \tag{6.3.32}$$

We first estimate (6.3.31) with Lemma 6.3.2 by

$$\leq \sum_{\mathfrak{p} \in \mathfrak{A}' : \mathcal{I}(\mathfrak{p}) = L'} \mu(E(\mathfrak{p}) \cap G \cap L') \leq 2^{a(N+5)} \operatorname{dens}_1(\mathfrak{A}) \mu(L'). \tag{6.3.33}$$

We turn to (6.3.32). Consider the element $\mathfrak{p}' \in \mathfrak{A}'$ as above with $L \subset \mathcal{I}(\mathfrak{p}')$ and $s(L) < s(\mathfrak{p}')$. As $L \subset L'$ and $s(L') = s(L) + 1$, we conclude with the dyadic property that $L' \subset \mathcal{I}(\mathfrak{p}')$. By maximality of L, we have $L' \notin \mathcal{L}$. This together with the existence of the given $\mathfrak{p}' \in \mathfrak{A}$ with $L' \subset \mathcal{I}(\mathfrak{p}')$ shows by definition of $\mathcal{L}$ that there exists $\mathfrak{p}'' \in \mathfrak{A}'$ with $\mathcal{I}(\mathfrak{p}'') \subset L'$.

By the covering property (2.0.13), there exists a unique $\mathfrak{p}_\vartheta$ with

$$\mathcal{I}(\mathfrak{p}_\vartheta) = L'$$

such that $\vartheta \in \Omega(\mathfrak{p}_\vartheta)$. Note that

$$\vartheta \in B(\mathcal{Q}(\mathfrak{p}_\vartheta), 1) \tag{6.3.34}$$

and as $\mathfrak{p}'' \in \mathfrak{A}_{\vartheta, N}$ that

$$\vartheta \in B(\mathcal{Q}(\mathfrak{p}''), 2^{N+1}). \tag{6.3.35}$$

By Lemma 6.3.1, we conclude

$$2^{N+3}\mathfrak{p}'' \lesssim 2^{N+3}\mathfrak{p}_\vartheta \,. \tag{6.3.36}$$

As $\mathfrak{p}'' \in \mathfrak{A}'$, we have by Definition (2.0.28) of dens_1 that

$$\mu(E_2(2^{N+3}, \mathfrak{p}_\vartheta)) \le 2^{Na+3a} \, \mathrm{dens}_1(\mathfrak{A})\mu(L') \,. \tag{6.3.37}$$

Now let $\mathfrak{p}$ be any tile in the summation set in (6.3.32), that is, $\mathfrak{p} \in \mathfrak{A}'$ and $\mathcal{J}(\mathfrak{p}) \neq L'$. Then $\mathcal{J}(\mathfrak{p}) \cap L \neq \varnothing$. It follows by the dyadic property (2.0.8) and the definition of L that $L \subset \mathcal{J}(\mathfrak{p})$ and $L \neq \mathcal{J}(\mathfrak{p})$. By the dyadic property (2.0.8), we have $s(L) < \mathrm{s}(\mathfrak{p})$ and thus $s(L') \le \mathrm{s}(\mathfrak{p})$. By the dyadic property (2.0.8) again, we have $L' \subset \mathcal{J}(\mathfrak{p})$. As $L' \neq \mathcal{J}(\mathfrak{p})$, we conclude $s(L) < \mathrm{s}(\mathfrak{p})$. By Lemma 6.3.3, we can thus estimate (6.3.32) by

$$\sum_{\mathfrak{p}\in\mathfrak{A}':\mathcal{J}(\mathfrak{p})\neq L'} \mu(E(\mathfrak{p}) \cap G \cap L') \le \mu(E_2(2^{N+3}, \mathfrak{p}_\vartheta)) \,. \tag{6.3.38}$$

Using the decomposition into (6.3.31) and (6.3.32) and the estimates (6.3.33), (6.3.21), (6.3.37) we obtain the estimate

$$\sum_{\mathfrak{p}\in\mathfrak{A}'} \mu(E(\mathfrak{p}) \cap G \cap L) \le (2^{a(N+5)} + 2^{Na+3a}) \, \mathrm{dens}_1(\mathfrak{A})\mu(L') \,. \tag{6.3.39}$$

Using $s(L') = s(L) + 1$ and $D = 2^{100a^2}$ and the squeezing property (2.0.10) and the doubling property (1.0.5) $100a^2 + 4$ times , we obtain

$$\mu(L') \le 2^{100a^3+4a}\mu(L) \,. \tag{6.3.40}$$

Inserting in (6.3.39) and using $a \ge 4$ gives (6.3.30). This completes the proof of the lemma. $\qquad\square$

We turn to the proof of Lemma 6.1.6.

PROOF OF LEMMA 6.1.6. Using that $\mathfrak{A}$ is the union of the $\mathfrak{A}_{\vartheta,N}$ with $N \ge 0$, we estimate the left-hand side (6.1.45) with the triangle inequality by

$$\le \sum_{N\ge 0} \left\| \sum_{\mathfrak{p}\in\mathfrak{A}_{\vartheta,N}} 2^{-N/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \right\|_p \tag{6.3.41}$$

We consider each individual term in this sum and estimate it's p-th power. Using that for each $x \in X$ by Lemma 6.3.4 there is at most one $\mathfrak{p} \in \mathfrak{A}$ with $x \in E(\mathfrak{p})$, we have

$$\left\| \sum_{\mathfrak{p}\in\mathfrak{A}_{\vartheta,N}} 2^{-N/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \right\|_p^p \tag{6.3.42}$$

$$= \int_G \left(\sum_{\mathfrak{p}\in\mathfrak{A}_{\vartheta,N}} 2^{-N/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})}(x) \right)^P d\mu(x) \tag{6.3.43}$$

$$= \int_G \sum_{\mathfrak{p}\in\mathfrak{A}_{\vartheta,N}} 2^{-pN/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})}(x) \, d\mu(x) \tag{6.3.44}$$

$$= 2^{-pN/(2a^2+a^3)} \sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}} \mu(E(\mathfrak{p}) \cap G) \tag{6.3.45}$$

Using Lemma 6.3.4, we estimate the last display by

$$\leq 2^{-pN/(2a^2+a^3)+101a^3+Na} \operatorname{dens}_1(\mathfrak{A}) \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{J}(\mathfrak{p}) \right) \tag{6.3.46}$$

Using that with $a \geq 4$ and since $p = 4a^4$, we have

$$pN/(2a^2 + a^3) \geq 4a^4 N/(3a^3) \geq Na + N \,. \tag{6.3.47}$$

Hence we have for (6.3.46) the upper bound

$$\leq 2^{101a^3-N} \operatorname{dens}_1(\mathfrak{A}) \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{J}(\mathfrak{p}) \right) \,. \tag{6.3.48}$$

Taking th p-th root and summing over $N \geq 0$ gives for (6.3.41) the upper bound

$$\leq \left(\sum_{N \geq 0} 2^{-N/p} \right) 2^{101a^3/p} \operatorname{dens}_1(\mathfrak{A})^{\frac{1}{p}} \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{J}(\mathfrak{p}) \right)^{\frac{1}{p}} \tag{6.3.49}$$

$$\leq \left(1 - 2^{-1/p} \right)^{-1} 2^{101a^3/p} \operatorname{dens}_1(\mathfrak{A})^{\frac{1}{p}} \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{J}(\mathfrak{p}) \right)^{\frac{1}{p}} \,. \tag{6.3.50}$$

Using that $p = 4a^4$ and $a \geq 4$, this proves the lemma. $\qquad\square$

Proof of the Forest Operator Proposition

7.1. The pointwise tree estimate

Fix a forest $(\mathfrak{U}, \mathfrak{T})$. The main result of this subsection is Lemma 7.1.3, we begin this section with some definitions necessary to state the lemma.

For $\mathfrak{u} \in \mathfrak{U}$ and $x \in X$, we define

$$\sigma(\mathfrak{u}, x) := \{\mathrm{s}(\mathfrak{p}) : \mathfrak{p} \in \mathfrak{T}(\mathfrak{u}), x \in E(\mathfrak{p})\} .$$

This is a subset of $\mathbb{Z} \cap [-S, S]$, so has a minimum and a maximum. We set

$$\overline{\sigma}(\mathfrak{u}, x) := \max \sigma(\mathfrak{T}(\mathfrak{u}), x)$$

$$\underline{\sigma}(\mathfrak{u}, x) := \min \sigma(\mathfrak{T}(\mathfrak{u}), x) .$$

LEMMA 7.1.1 (convex scales). *For each $\mathfrak{u} \in \mathfrak{U}$, we have*

$$\sigma(\mathfrak{u}, x) = \mathbb{Z} \cap [\underline{\sigma}(\mathfrak{u}, x), \overline{\sigma}(\mathfrak{u}, x)] .$$

PROOF. Let $s \in \mathbb{Z}$ with $\underline{\sigma}(\mathfrak{u}, x) \leq s \leq \overline{\sigma}(\mathfrak{u}, x)$. By definition of σ, there exists $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $\mathrm{s}(\mathfrak{p}) = \underline{\sigma}(\mathfrak{u}, x)$ and $x \in E(\mathfrak{p})$, and there exists $\mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u})$ with $\mathrm{s}(\mathfrak{p}'') = \overline{\sigma}(\mathfrak{u}, x)$ and $x \in E(\mathfrak{p}'') \subset \mathcal{I}(\mathfrak{p}'')$. By property (2.0.7) of the dyadic grid, there exists a cube $I \in \mathcal{D}$ of scale s with $x \in I$. By property (2.0.13), there exists a tile $\mathfrak{p}' \in \mathfrak{P}(I)$ with $Q(x) \in \Omega(\mathfrak{p}')$. By the dyadic property (2.0.8) we have $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}'')$, and by (2.0.14), we have $\Omega(\mathfrak{p}'') \subset \Omega(\mathfrak{p}') \subset \Omega(\mathfrak{p})$. Thus $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$, which gives with the convexity property (2.0.33) of $\mathfrak{T}(\mathfrak{u})$ that $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u})$, so $s \in \sigma(\mathfrak{u}, x)$. $\qquad\square$

For a nonempty collection of tiles $\mathfrak{S} \subset \mathfrak{P}$ we define

$$\mathcal{J}_0(\mathfrak{S})$$

to be the collection of all dyadic cubes $J \in \mathcal{D}$ such that $s(J) = -S$ or

$$\mathcal{I}(\mathfrak{p}) \not\subset B(c(J), 100D^{s(J)+1})$$

for all $\mathfrak{p} \in \mathfrak{S}$. We define $\mathcal{J}(\mathfrak{S})$ to be the collection of inclusion maximal cubes in $\mathcal{J}_0(\mathfrak{S})$.

We further define

$$\mathcal{L}_0(\mathfrak{S})$$

to be the collection of dyadic cubes $L \in \mathcal{D}$ such that $s(L) = -S$, or there exists $\mathfrak{p} \in \mathfrak{S}$ with $L \subset \mathcal{I}(\mathfrak{p})$ and there exists no $\mathfrak{p} \in \mathfrak{S}$ with $\mathcal{I}(\mathfrak{p}) \subset L$. We define $\mathcal{L}(\mathfrak{S})$ to be the collection of inclusion maximal cubes in $\mathcal{L}_0(\mathfrak{S})$.

Lemma 7.1.2 (dyadic partitions). *For each $\mathfrak{S} \subset \mathfrak{P}$, we have*

$$\bigcup_{I \in \mathcal{D}} I = \dot{\bigcup_{J \in \mathcal{J}(\mathfrak{S})}} J \tag{7.1.1}$$

and

$$\bigcup_{\mathfrak{p} \in \mathfrak{S}} \mathcal{J}(\mathfrak{p}) = \dot{\bigcup_{L \in \mathcal{L}(\mathfrak{S})}} L. \tag{7.1.2}$$

Proof. Since $\mathcal{J}(\mathfrak{S})$ is the set of inclusion maximal cubes in $\mathcal{J}_0(\mathfrak{S})$, cubes in $\mathcal{J}(\mathfrak{S})$ are pairwise disjoint by (2.0.8). The same applies to $\mathcal{L}(\mathfrak{S})$.

If $x \in \bigcup_{I \in \mathcal{D}} I$, then there exists by (2.0.7) a cube $I \in \mathcal{D}$ with $x \in I$ and $s(I) = -S$. Then $I \in \mathcal{J}_0(\mathfrak{S})$. There exists an inclusion maximal cube in $\mathcal{J}_0(\mathfrak{S})$ containing I. This cube contains x and is contained in $\mathcal{J}(\mathfrak{S})$. This shows one inclusion in (7.1.1), the other one follows from $\mathcal{J}(\mathfrak{S}) \subset \mathcal{D}$.

The proof of the two inclusions in (7.1.2) is similar. □

For a finite collection of pairwise disjoint cubes $\mathcal{C}$, define the projection operator

$$P_{\mathcal{C}} f(x) := \sum_{J \in \mathcal{C}} \mathbf{1}_J(x) \frac{1}{\mu(J)} \int_J f(y) \, \mathrm{d}\mu(y).$$

Given a scale $-S \le s \le S$ and a point $x \in \bigcup_{I \in \mathcal{D}, s(I)=s} I$, there exists a unique cube in $\mathcal{D}$ of scale s containing x by (2.0.7). We denote it by $I_s(x)$. Define for $\vartheta \in \Theta$ the nontangential maximal operator

$$T_{\mathcal{N}}^{\vartheta} f(x) := \sup_{-S \le s_1 < S} \sup_{x' \in I_{s_1}(x)} \sup_{\substack{s_1 < s_2 \le S \\ D^{s_2 - 1} \le R_Q(\vartheta, x')}} \left| \sum_{s=s_1}^{s_2} \int K_s(x', y) f(y) \, \mathrm{d}\mu(y) \right|. \tag{7.1.3}$$

Define for each $\mathfrak{u} \in \mathfrak{U}$ the auxiliary operator

$$S_{1,\mathfrak{u}} f(x)$$

$$:= \sum_{I \in \mathcal{D}} \mathbf{1}_I(x) \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(c(I), 16D^{s(I)}) \\ s(J) \le s(I)}} \frac{D^{(s(J)-s(I))/a}}{\mu(B(c(I), 16D^{s(I)}))} \int_J |f(y)| \, \mathrm{d}\mu(y). \tag{7.1.4}$$

Define also the collection of balls

$$\mathcal{B} = \{B(c(I), 2^s D^{s(I)}) \; : \; I \in \mathcal{D}, 0 \le s \le S+5\}.$$

The following pointwise estimate for operators associated to sets $\mathfrak{T}(\mathfrak{u})$ is the main result of this subsection.

Lemma 7.1.3 (pointwise tree estimate). *Let $\mathfrak{u} \in \mathfrak{U}$ and $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$. Let $x, x' \in L$. Then for all bounded functions f with bounded support*

$$\left| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}[e(-Q(\mathfrak{u}))f](x) \right|$$

$$\le 2^{151a^3}(M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x') + |T_{\mathcal{N}}^{Q(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(x')|, \tag{7.1.5}$$

PROOF. By (2.0.21), if $T_{\mathfrak{p}}[e(-\mathcal{Q}(\mathfrak{u}))f](x) \neq 0$, then $x \in E(\mathfrak{p})$. Combining this with $|e(\mathcal{Q}(\mathfrak{u})(x) - Q(x)(x))| = 1$, we obtain

$$| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}[e(-\mathcal{Q}(\mathfrak{u}))f](x)|$$

$$= \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int e(-\mathcal{Q}(\mathfrak{u})(y) + Q(x)(y) + \mathcal{Q}(\mathfrak{u})(x) - Q(x)(x)) \times \right.$$

$$\left. K_s(x,y)f(y)\,\mathrm{d}\mu(y) \right|.$$

Using the triangle inequality, we bound this by the sum of three terms:

$$\leq \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int (e(-\mathcal{Q}(\mathfrak{u})(y) + Q(x)(y) + \mathcal{Q}(\mathfrak{u})(x) - Q(x)(x)) - 1) \times \right.$$

$$\left. K_s(x,y)f(y)\,\mathrm{d}\mu(y) \right| \quad (7.1.6)$$

$$+ \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int K_s(x,y) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}f(y)\,\mathrm{d}\mu(y) \right| \quad (7.1.7)$$

$$+ \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int K_s(x,y)(f(y) - P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}f(y))\,\mathrm{d}\mu(y) \right|. \quad (7.1.8)$$

The proof is completed using the bounds for these three terms proven in Lemma 7.1.4, Lemma 7.1.5 and Lemma 7.1.6. □

LEMMA 7.1.4 (first tree pointwise). *For all $\mathfrak{u} \in \mathfrak{U}$, all $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, all $x, x' \in L$ and all bounded f with bounded support, we have*

$$(7.1.6) \leq 10 \cdot 2^{105a^3} M_{\mathcal{B},1} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|(x').$$

PROOF. Let $s \in \sigma(\mathfrak{u}, x)$. If $x, y \in X$ are such that $K_s(x,y) \neq 0$, then, by (2.1.2), we have $\rho(x,y) \leq 1/2D^s$. By 1-Lipschitz continuity of the function $t \mapsto \exp(it) = e(t)$ and the property (1.0.7) of the metrics d_B, it follows that

$$|e(-\mathcal{Q}(\mathfrak{u})(y) + Q(x)(y) + \mathcal{Q}(\mathfrak{u})(x) - Q(x)(x)) - 1|$$

$$\leq d_{B(x,1/2D^s)}(\mathcal{Q}(\mathfrak{u}), Q(x)).$$

Let $\mathfrak{p}_s \in \mathfrak{T}(\mathfrak{u})$ be a tile with $\mathrm{s}(\mathfrak{p}_s) = s$ and $x \in E(\mathfrak{p}_s)$, and let $\mathfrak{p}'$ be a tile with $\mathrm{s}(\mathfrak{p}') = \overline{\sigma}(\mathfrak{u},x)$ and $x \in E(\mathfrak{p}')$. Using the doubling property (1.0.8), the definition of $d_{\mathfrak{p}}$ and Lemma 2.1.2, we can bound the previous display by

$$2^a d_{\mathfrak{p}_s}(\mathcal{Q}(\mathfrak{u}), Q(x)) \leq 2^a 2^{s - \overline{\sigma}(\mathfrak{u},x)} d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}), Q(x)).$$

Since $\mathcal{Q}(\mathfrak{u}) \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 4)$ by (2.0.32) and $Q(x) \in \Omega(\mathfrak{p}') \subset B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 1)$ by (2.0.15), this is estimated by

$$\leq 5 \cdot 2^a 2^{s - \overline{\sigma}(\mathfrak{u},x)}.$$

Using (2.1.3), it follows that

$$(7.1.6) \leq 5 \cdot 2^{103a^3} \sum_{s \in \sigma(x)} 2^{s - \overline{\sigma}(\mathfrak{u}, x)} \frac{1}{\mu(B(x, D^s))} \int_{B(x, 0.5D^s)} |f(y)| \, d\mu(y) \, .$$

By (7.1.1), the collection $\mathcal{J}$ is a partition of X, so this is estimated by

$$5 \cdot 2^{103a^3} \sum_{s \in \sigma(x)} 2^{s - \overline{\sigma}(\mathfrak{u}, x)} \frac{1}{\mu(B(x, D^s))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \cap B(x, 0.5D^s) \neq \varnothing}} \int_J |f(y)| \, d\mu(y) \, .$$

This expression does not change if we replace $|f|$ by $P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|$.

Let $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ with $B(x, 0.5D^s) \cap J \neq \varnothing$. By the triangle inequality and since $x \in E(\mathfrak{p}_s) \subset B(c(\mathfrak{p}_s), 4D^s)$, it follows that $B(c(\mathfrak{p}_s), 4.5D^s) \cap J \neq \varnothing$. If $s(J) \geq s$ and $s(J) > -S$, then it follows from the triangle inequality, (2.0.10) and (2.0.1) that $\mathcal{I}(\mathfrak{p}_s) \subset B(c(J), 100D^{s(J)+1})$, contradicting $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$. Thus $s(J) \leq s - 1$ or $s(J) = -S$. If $s(J) = -S$ and $s(J) > s - 1$, then $s = -S$. Thus we always have $s(J) \leq s$. It then follows from the triangle inequality and (2.0.10) that $J \subset B(c(\mathfrak{p}_s), 16D^s)$.

Thus we can continue our chain of estimates with

$$5 \cdot 2^{103a^3} \sum_{s \in \sigma(x)} 2^{s - \overline{\sigma}(\mathfrak{u}, x)} \frac{1}{\mu(B(x, D^s))} \int_{B(c(\mathfrak{p}_s), 16D^s)} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f(y)| \, d\mu(y) \, .$$

We have $B(c(\mathfrak{p}_s), 16D^s)) \subset B(x, 32D^s)$, by (2.0.10) and the triangle inequality, since $x \in \mathcal{I}(\mathfrak{p})$. Combining this with the doubling property (1.0.5), we obtain

$$\mu(B(c(\mathfrak{p}_s), 16D^s)) \leq 2^{5a} \mu(B(x, D^s)) \, .$$

Since $a \geq 4$, it follows that (7.1.6) is bounded by

$$2^{104a^3} \sum_{s \in \sigma(x)} 2^{s - \overline{\sigma}(\mathfrak{u}, x)} \frac{1}{\mu(B(c(\mathfrak{p}_s), 16D^s))} \int_{B(c(\mathfrak{p}_s), 16D^s)} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f(y)| \, d\mu(y) \, .$$

Since $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, we have $s(L) \leq s(\mathfrak{p})$ for all $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$. Since $x \in L \cap \mathcal{I}(\mathfrak{p}_s)$, it follows by (2.0.8) that $L \subset \mathcal{I}(\mathfrak{p}_s)$, in particular $x' \in \mathcal{I}(\mathfrak{p}_s) \subset B(c(\mathfrak{p}_s), 16D^s)$. Thus

$$\leq 2^{104a^3} \sum_{s \in \sigma(x)} 2^{s - \overline{\sigma}(\mathfrak{u}, x)} M_{\mathcal{B}, 1} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x')$$

$$\leq 2^{105a^3} M_{\mathcal{B}, 1} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x') \, .$$

This completes the estimate for term (7.1.6). $\qquad \square$

LEMMA 7.1.5 (second tree pointwise). *For all $\mathfrak{u} \in \mathfrak{U}$, all $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, all $x, x' \in L$ and all bounded f with bounded support, we have*

$$\left| \sum_{s \in \sigma(\mathfrak{u}, x)} \int K_s(x, y) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(y) \, d\mu(y) \right| \leq T_{\mathcal{N}}^{Q(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(x') \, .$$

PROOF. Let $s_1 = \underline{\sigma}(\mathfrak{u}, x)$. By definition, there exists a tile $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $s(\mathfrak{p}) = s_1$ and $x \in E(\mathfrak{p})$. Then $x \in \mathcal{I}(\mathfrak{p}) \cap L$. By (2.0.8) and the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, it follows that $L \subset \mathcal{I}(\mathfrak{p})$, in particular $x' \in \mathcal{I}(\mathfrak{p})$, so $x \in I_{s_1}(x')$. Next, let

$s_2 = \overline{\sigma}(\mathfrak{u}, x)$ and let $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u})$ with $s(\mathfrak{p}') = s_2$ and $x \in E(\mathfrak{p}')$. Since $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u})$, we have $4\mathfrak{p}' \lesssim 1\mathfrak{u}$. Since $Q(x) \in \Omega(\mathfrak{p}')$, it follows that

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}), Q(x)) \le 5 \,.$$

Applying the doubling property (1.0.8) five times, we obtain

$$d_{B(c(\mathfrak{p}), 8D^{s_2})}(\mathcal{Q}(\mathfrak{u}), Q(x)) \le 5 \cdot 2^{5a} \,.$$

By the triangle inequality, we have $B(x, D^{s_2}) \subset B(c(\mathfrak{p}), 8D^{s_2})$, so by (1.0.9)

$$d_{B(x, D^{s_2})}(\mathcal{Q}(\mathfrak{u}), Q(x)) \le 5 \cdot 2^{5a} \,.$$

Finally, by applying (1.0.10) 2^{100a} times, we obtain

$$d_{B(x, D^{s_2-1})}(\mathcal{Q}(\mathfrak{u}), Q(x)) \le 5 \cdot 2^{-95a} < 1 \,.$$

Consequently, $D^{s_2-1} < R_Q(\mathcal{Q}(\mathfrak{u}), x)$. The lemma now follows from the definition of $T_{\mathcal{N}}$. $\qquad\square$

LEMMA 7.1.6 (third tree pointwise). *For all $\mathfrak{u} \in \mathfrak{U}$, all $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, all $x, x' \in L$ and all bounded f with bounded support, we have*

$$\left| \sum_{s \in \sigma(\mathfrak{u}, x)} \int K_s(x, y)(f(y) - P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}f(y)) \, \mathrm{d}\mu(y) \right|$$

$$\le 2^{151a^3} S_{1,\mathfrak{u}} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|(x') \,.$$

PROOF. We have for $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$:

$$\int_J K_s(x, y)(1 - P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))})f(y) \, \mathrm{d}\mu(y)$$

$$= \int_J \frac{1}{\mu(J)} \int_J K_s(x, y) - K_s(x, z) \, \mathrm{d}\mu(z) \, f(y) \, \mathrm{d}\mu(y) \,. \tag{7.1.9}$$

By (2.1.4) and (2.0.10), we have for $y, z \in J$

$$|K_s(x, y) - K_s(x, z)| \le \frac{2^{150a^3}}{\mu(B(x, D^s))} \left(\frac{8D^{s(J)}}{D^s} \right)^{1/a} \,.$$

Suppose that $s \in \sigma(\mathfrak{u}, x)$. If $K_s(x, y) \ne 0$ for some $y \in J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ then, by (2.1.2), $y \in B(x, 0.5D^s) \cap J \ne \varnothing$. Let $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $s(\mathfrak{p}) = s$ and $x \in E(\mathfrak{p})$. Then $B(c(\mathfrak{p}_s), 4.5D^s) \cap J \ne \varnothing$ by the triangle inequality. If $s(J) \ge s$ and $s(J) > -S$, then it follows from the triangle inequality, (2.0.10) and (2.0.1) that $\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1})$, contradicting $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$. Thus $s(J) \le s-1$ or $s(J) = -S$. If $s(J) = -S$ and $s(J) > s-1$, then $s = -S$. So in both cases, $s(J) \le s$. It then follows from the triangle inequality and (2.0.10) that $J \subset B(x, 16D^s)$.

Thus, we can estimate (7.1.8) by

$$2^{150a^3+3/a} \sum_{\mathfrak{p} \in \mathfrak{T}} \frac{\mathbf{1}_{E(\mathfrak{p})}(x)}{\mu(B(x, D^{s(\mathfrak{p})}))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(x, 16D^{s(\mathfrak{p})}) \\ s(J) \le s(\mathfrak{p})}} D^{(s(J)-s(\mathfrak{p}))/a} \int_J |f| \,.$$

$$= 2^{150a^3+3/a} \sum_{I \in \mathcal{D}} \sum_{\substack{\mathfrak{p} \in \mathfrak{T} \\ \mathcal{J}(\mathfrak{p})=I}} \frac{\mathbf{1}_{E(\mathfrak{p})}(x)}{\mu(B(x, D^{s(I)}))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(x, 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J)-s(I))/a} \int_J |f| \,.$$

By (2.0.13) and (2.0.20), the sets $E(\mathfrak{p})$ for tiles $\mathfrak{p}$ with $\mathcal{J}(\mathfrak{p}) = I$ are pairwise disjoint. It follows from the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ that $x \in \mathcal{J}(\mathfrak{p})$ if and only if $x' \in \mathcal{J}(\mathfrak{p})$, thus we can estimate the sum over $\mathbf{1}_{E(\mathfrak{p})}(x)$ by $\mathbf{1}_I(x')$. If $x \in E(\mathfrak{p})$ then in particular $x \in \mathcal{J}(\mathfrak{p})$, so by (2.0.10) $B(c(I), 16D^{s(I)}) \subset B(x, 32D^{s(I)})$. By the doubling property (1.0.5)

$$\mu(B(c(I), 16D^{s(I)})) \leq 2^{5a}\mu(B(x, D^{s(I)})) \,.$$

Since $a \geq 4$ we can continue our estimate with

$$\leq 2^{151a^3} \sum_{I \in \mathcal{D}} \frac{\mathbf{1}_I(x')}{\mu(B(c(I), 16D^{s(I)}))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(x, 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J)-s(I))/a} \int_J |f|$$

$$= 2^{151a^3} S_{1,\mathfrak{u}} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x') \,.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.2. An auxiliary L^2 tree estimate

In this subsection we prove the following estimate on L^2 for operators associated to trees.

LEMMA 7.2.1 (tree projection estimate). *Let $\mathfrak{u} \in \mathfrak{U}$. Then we have for all f, g bounded with bounded support*

$$\left| \int_X \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \bar{g}(y) T_\mathfrak{p} f(y) \,\mathrm{d}\mu(y) \right|$$

$$\leq 2^{104a^3} \|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2 \|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|\|_2. \qquad\qquad (7.2.1)$$

Below, we deduce Lemma 7.2.1 from Lemma 7.1.3 and the following estimates for the operators in Lemma 7.1.3.

LEMMA 7.2.2 (nontangential operator bound). *For all bounded f with bounded supportand all $\vartheta \in \Theta$*

$$\|T_\mathcal{N}^\vartheta f\|_2 \leq 2^{103a^3}\|f\|_2 \,.$$

LEMMA 7.2.3 (boundary operator bound). *For all $\mathfrak{u} \in \mathfrak{U}$ and all bounded functions f with bounded support*

$$\|S_{1,\mathfrak{u}}f\|_2 \leq 2^{12a}\|f\|_2 \,. \qquad\qquad (7.2.2)$$

PROOF OF LEMMA 7.2.1. Let $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$. We apply Lemma 7.1.3 to $e(Q(\mathfrak{u}))f$ to obtain for all $y, x' \in L$

$$\left| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_\mathfrak{p} f(y) \right|$$

$$\leq 2^{151a^3}((M_{\mathcal{B},1} + S_{1,\mathfrak{u}})P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|(x') + |T_\mathcal{N}^{Q(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}[e(Q(\mathfrak{u}))f](x')|) \,.$$

Hence, by taking an infimum, we have for $y \in L$

$$\left| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y) \right|$$

$$\leq 2^{151a^3} \inf_{x' \in L} \left((M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x') + |T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} [e(\mathcal{Q}(\mathfrak{u})) f](x')| \right).$$

Integrating this estimate yields

$$\int_L |g(y)| \left| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y) \right| \mathrm{d}\mu(y)$$

$$\leq 2^{151a^3} \int_L |g(y)| \inf_{x' \in L} \left((M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x') + |T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} [e(\mathcal{Q}(\mathfrak{u})) f](x')| \right) \mathrm{d}\mu(y)$$

$$\leq 2^{151a^3} \int_L |g(y)| \, \mathrm{d}\mu(y) \times$$
$$\int_L 2 \left((M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(y) + |T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} [e(\mathcal{Q}(\mathfrak{u})) f](y)| \right) \mathrm{d}\mu(y)$$

$$= 2^{151a^3} \int_L P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} |g|(y) \times$$
$$\left((M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(y) + |T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} [e(\mathcal{Q}(\mathfrak{u})) f](y)| \right) \mathrm{d}\mu(y).$$

By (2.0.21), we have $T_{\mathfrak{p}} f = \mathbf{1}_{\mathcal{J}(\mathfrak{p})} T_{\mathfrak{p}} f$ for all $\mathfrak{p} \in \mathfrak{P}$, so

$$\left| \int \bar{g}(y) \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y) \, \mathrm{d}\mu(y) \right| = \left| \int_{\bigcup_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \mathcal{J}(\mathfrak{p})} \bar{g}(y) \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y) \, \mathrm{d}\mu(y) \right|.$$

Since $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ partitions $\bigcup_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \mathcal{J}(\mathfrak{p})$ by Lemma 7.1.2, we get from the triangle inequality

$$\leq \sum_{L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))} \int_L |g(y)| \left| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y) \right| \mathrm{d}\mu(y)$$

which by the above computation is bounded by

$$2^{151a^3} \sum_{L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))} \int_L P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} |g|(y) \times$$
$$\left((M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(y) + |T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} [e(\mathcal{Q}(\mathfrak{u})) f](y)| \right) \mathrm{d}\mu(y)$$

$$= 2^{151a^3} \int_X P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} |g|(y) \times$$
$$\left((M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(y) + |T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} [e(\mathcal{Q}(\mathfrak{u})) f](y)| \right) \mathrm{d}\mu(y).$$

Applying Cauchy-Schwarz and Minkowski's inequality, this is bounded by

$$2^{151a^3}\|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|\|_2 \times$$

$$(\|M_{\mathcal{B},1}P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2 + \|S_{1,\mathfrak{u}}P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2 + \|T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})}P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}[e(\mathcal{Q}(\mathfrak{u}))f](y)\|_2).$$

By Proposition 2.0.6, Lemma 7.2.2 and Lemma 7.2.3, the second factor is at most

$$(2^{2a+1} + 2^{12a})\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2 + 2^{103a^3}\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}[e(\mathcal{Q}(\mathfrak{u}))f]\|_2.$$

By the triangle inequality we have for all $x \in X$ that $|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}[e(\mathcal{Q}(\mathfrak{u}))f]|(x) \le P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|(x)$, thus we can further estimate the above by

$$(2^{2a+1} + 2^{12a} + 2^{103a^3})\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2.$$

This completes the proof since $a \ge 4$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we prove the two auxiliary lemmas. We begin with the nontangential maximal operator $T_{\mathcal{N}}$.

PROOF OF LEMMA 7.2.2. Fix s_1, s_2. By (2.0.4) we have for all $x \in (0, \infty)$

$$\sum_{s=s_1}^{s_2} \psi(D^{-s}x) = 1 - \sum_{s<s_1} \psi(D^{-s}x) - \sum_{s>s_1} \psi(D^{-s}x).$$

Since ψ is supported in $[\frac{1}{4D}, \frac{1}{2}]$, the two sums on the right hand side are zero for all $x \in [\frac{1}{2}D^{s_1-1}, \frac{1}{4}D^{s_2-1}]$, hence

$$x \in [\frac{1}{2}D^{s_1-1}, \frac{1}{4}D^{s_2}] \implies \sum_{s=s_1}^{s_2} \psi(D^{-s}x) = 1.$$

Since ψ is supported in $[\frac{1}{4D}, \frac{1}{2}]$, we further have

$$x \notin [\frac{1}{4}D^{s_1-1}, \frac{1}{2}D^{s_2}] \implies \sum_{s=s_1}^{s_2} \psi(D^{-s}x) = 0.$$

Finally, since $\psi \ge 0$ and $\sum_{s\in\mathbb{Z}} \psi(D^{-s}x) = 1$, we have for all x

$$0 \le \sum_{s=s_1}^{s_2} \psi(D^{-s}x) \le 1.$$

Let $x' \in I_{s_1}(x)$ and suppose that $D^{s_2-1} \le R_Q(\vartheta, x')$. By the triangle inequality and (2.0.10), it holds that $\rho(x, x') \le 8D^{s_1}$. We have

$$\left| \sum_{s=s_1}^{s_2} \int K_s(x', y)f(y)\, \mathrm{d}\mu(y) \right|$$

$$= \left| \int \sum_{s=s_1}^{s_2} \psi(D^{-s}\rho(x', y))K(x', y)f(y)\, \mathrm{d}\mu(y) \right|$$

$$\le \left| \int_{8D^{s_1} \le \rho(x',y) \le \frac{1}{4}D^{s_2-1}} K(x', y)f(y)\, \mathrm{d}\mu(y) \right| \qquad\qquad (7.2.3)$$

$$+ \int_{\frac{1}{4}D^{s_1-1} \leq \rho(x',y) \leq 8D^{s_1}} |K(x',y)||f(y)| \, d\mu(y) \tag{7.2.4}$$

$$+ \int_{\frac{1}{4}D^{s_2-1} \leq \rho(x',y) \leq \frac{1}{2}D^{s_2}} |K(x',y)||f(y)| \, d\mu(y) \,. \tag{7.2.5}$$

The first term (7.2.3) is at most $2T_Q^\vartheta f(x)$, using with $R_1 := 8D^{s_1}$ and $R_2 := \frac{1}{4}D^{s_2-1}$ and

$$R_1 \leq R_2 < R_Q(\vartheta, x') \tag{7.2.6}$$

the triangle inequality in the form

$$\sup_{\rho(x,x')<R_1} \left| \int_{R_1<\rho(x',y)<R_2} K(x',y)f(y) \, d\mu(y) \right| \tag{7.2.7}$$

$$\leq \sup_{\rho(x,x')<R_1} \left| \int_{R_1<\rho(x',y)<R_Q(\vartheta,x')} K(x',y)f(y) \, d\mu(y) \right| \tag{7.2.8}$$

$$+ \sup_{\rho(x,x')<R_1} \left| \int_{R_2<\rho(x',y)<R_Q(\vartheta,x')} K(x',y)f(y) \, d\mu(y) \right| \tag{7.2.9}$$

and making the second term larger by replacing the constraint $\rho(x,x') < R_1$ by the constraint $\rho(x,x') < R_2$.

The other two terms will be estimated by the finitary maximal function from Proposition 2.0.6. For the second term (7.2.4) we use (1.0.14) which implies that for all y with $\rho(x',y) \geq \frac{1}{4}D^{s_1-1}$, we have

$$|K(x',y)| \leq \frac{2^{a^3}}{\mu(B(x', \frac{1}{4}D^{s_1-1}))} \,.$$

Using $D = 2^{100a^2}$ and the doubling property (1.0.5) $6 + 100a^2$ times estimates the last display by

$$\leq \frac{2^{6a+101a^3}}{\mu(B(x', 16D^{s_1}))} \,. \tag{7.2.10}$$

By the triangle inequality and (2.0.10), we have

$$B(x', 8D^{s_1}) \subset B(c(I_{s_1}(x)), 16D^{s(I_{s_1}(x))}) \,.$$

Combining this with (7.2.10), we conclude that (7.2.4) is at most

$$2^{6a+101a^3} M_{\mathcal{B},1} f(x) \,.$$

For (7.2.5) we argue similarly. We have for all y with $\rho(x',y) \geq \frac{1}{4}D^{s_2}$

$$|K(x',y)| \leq \frac{2^{a^3}}{\mu(B(x', \frac{1}{4}D^{s_2}))} \,.$$

Using the doubling property (1.0.5) $6 + 100a^2$ times estimates the last display by

$$\leq \frac{2^{6a+101a^3}}{\mu(B(x', 16D^{s_2}))} \,. \tag{7.2.11}$$

Note that by (2.0.8) we have $I_{s_1}(x) \subset I_{s_2}(x)$, in particular $x' \in I_{s_2}(x)$. By the triangle inequality and (2.0.10), we have

$$B(x', 8D^{s_2}) \subset B(c(I_{s_2}(x)), 16D^{s(I_{s_2}(x))}).$$

Combining this, (7.2.5) is at most

$$2^{6a+101a^3} M_{\mathcal{B},1} f(x).$$

Using $a \geq 4$, taking a supremum over all $x' \in I_{s_1}(x)$ and then a supremum over all $-S \leq s_1 < s_2 \leq S$, we obtain

$$T_{\mathcal{N}} f(x) \leq T_* f(x) + 2^{102a^3} M_{\mathcal{B},1} f(x).$$

The lemma now follows from assumption (1.0.18), Proposition 2.0.6 and $a \geq 4$. $\square$

We need the following lemma to prepare the L^2-estimate for the auxiliary operators $S_{1,\mathfrak{u}}$.

LEMMA 7.2.4 (boundary overlap). *For every cube $I \in \mathcal{D}$, there exist at most 2^{9a} cubes $J \in \mathcal{D}$ with $s(J) = s(I)$ and $B(c(I), 16D^{s(I)}) \cap B(c(J), 16D^{s(J)}) \neq \varnothing$.*

PROOF. Suppose that $B(c(I), 16D^{s(I)}) \cap B(c(J), 16D^{s(J)}) \neq \varnothing$ and $s(I) = s(J)$. Then $B(c(I), 33D^{s(I)}) \subset B(c(J), 128D^{s(J)})$. Hence by the doubling property (1.0.5)

$$2^{9a} \mu(B(c(J), \frac{1}{4} D^{s(J)})) \geq \mu(B(c(I), 33D^{s(I)})),$$

and by the triangle inequality, the ball $B(c(J), \frac{1}{4} D^{s(J)})$ is contained in $B(c(I), 33D^{s(I)})$.
If $\mathcal{C}$ is any finite collection of cubes $J \in \mathcal{D}$ satisfying $s(J) = s(I)$ and

$$B(c(I), 16D^{s(I)}) \cap B(c(J), 16D^{s(J)}) \neq \varnothing,$$

then it follows from (2.0.10) and pairwise disjointness of cubes of the same scale (2.0.8) that the balls $B(c(J), \frac{1}{4} D^{s(J)})$ are pairwise disjoint. Hence

$$\mu(B(c(I), 33D^{s(I)})) \geq \sum_{J \in \mathcal{C}} \mu(B(c(J), \frac{1}{4} D^{s(J)}))$$

$$\geq |\mathcal{C}| 2^{-9a} \mu(B(c(I), 33D^{s(I)})).$$

Since μ is doubling and $\mu \neq 0$, we have $\mu(B(c(I), 33D^{s(I)})) > 0$. The lemma follows after dividing by $2^{-9a} \mu(B(c(I), 33D^{s(I)}))$. $\square$

Now we can bound the operators $S_{1,\mathfrak{u}}$.

PROOF OF LEMMA 7.2.3. Note that by definition, $S_{1,\mathfrak{u}} f$ is a finite sum of indicator functions of cubes $I \in \mathcal{D}$ for each locally integrable f, and hence is bounded, has bounded support and is integrable. Let g be another function with the same three properties. Then $\bar{g} S_{1,\mathfrak{u}} f$ is integrable, and we have

$$\left| \int \bar{g}(y) S_{1,\mathfrak{u}} f(y) \, d\mu(y) \right|$$

$$= \left| \sum_{I \in \mathcal{D}} \frac{1}{\mu(B(c(I), 16D^{s(I)}))} \int_I \bar{g}(y) \, \mathrm{d}\mu(y) \right.$$
$$\left. \times \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) : J \subseteq B(c(I), 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J)-s(I))/a} \int_J |f(y)| \, \mathrm{d}\mu(y) \right|$$

$$\leq \sum_{I \in \mathcal{D}} \frac{1}{\mu(B(c(I), 16D^{s(I)}))} \int_{B(c(I), 16D^{s(I)})} |g(y)| \, \mathrm{d}\mu(y)$$
$$\times \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) : J \subseteq B(c(I), 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J)-s(I))/a} \int_J |f(y)| \, \mathrm{d}\mu(y) \,.$$

Changing the order of summation and using $J \subset B(c(I), 16D^{s(I)})$ to bound the first average integral by $M_{\mathcal{B},1}|g|(y)$ for any $y \in J$, we obtain

$$\leq \sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_J |f(y)| M_{\mathcal{B},1}|g|(y) \, \mathrm{d}\mu(y) \sum_{\substack{I \in \mathcal{D} : J \subset B(c(I), 16D^{s(I)}) \\ s(I) \geq s(J)}} D^{(s(J)-s(I))/a} \,. \quad (7.2.12)$$

By Lemma 7.2.4, there are at most 2^{9a} cubes I at each scale with $J \subset B(c(I), D^{s(I)})$. By convexity of $t \mapsto D^t$ and since $D \geq 2$, we have for all $-1 \leq t \leq 0$

$$D^t \leq 1 + t\left(1 - \frac{1}{D}\right) \leq 1 + \frac{1}{2}t \,,$$

so $(1 - D^{-1/a})^{-1} \leq 2a \leq 2^{a-1}$. Using this estimate for the sum of the geometric series, we conclude that (7.2.12) is at most

$$2^{10a-1} \sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_J |f(y)| M_{\mathcal{B},1}|g|(y) \, \mathrm{d}\mu(y) \,.$$

The collection $\mathcal{J}$ is a partition of X, so this equals

$$2^{11a-1} \int_X |f(y)| M_{\mathcal{B},1}|g|(y) \, \mathrm{d}\mu(y) \,.$$

Using Cauchy-Schwarz and Proposition 2.0.6, we conclude

$$\left| \int \bar{g} S_{1,\mathfrak{u}} f \, \mathrm{d}\mu \right| \leq 2^{12a} \|g\|_2 \|f\|_2 \,.$$

The lemma now follows by choosing $g = S_{1,\mathfrak{u}} f$ and dividing on both sides by the finite $\|S_{1,\mathfrak{u}} f\|_2$. $\qquad \square$

7.3. The quantitative L^2 tree estimate

The main result of this subsection is the following quantitative bound for operators associated to trees, with decay in the densities dens_1 and dens_2.

LEMMA 7.3.1 (densities tree bound). *Let $\mathfrak{u} \in \mathfrak{U}$. Then for all f, g bounded with bounded support*

$$\left| \int_X \bar{g} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \, d\mu \right| \leq 2^{155a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|f\|_2 \|g\|_2 \,. \tag{7.3.1}$$

If $|f| \leq \mathbf{1}_F$, then we have

$$\left| \int_X \bar{g} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \, d\mu \right| \leq 2^{256a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \operatorname{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \|f\|_2 \|g\|_2 \,. \tag{7.3.2}$$

Below, we deduce this lemma from Lemma 7.2.1 and the following two estimates controlling the size of support of the operator and its adjoint.

LEMMA 7.3.2 (local dens1 tree bound). *Let $\mathfrak{u} \in \mathfrak{U}$ and $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$. Then*

$$\mu(L \cap \bigcup_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{p})) \leq 2^{101a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u})) \mu(L) \,. \tag{7.3.3}$$

LEMMA 7.3.3 (local dens2 tree bound). *Let $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ be such that there exist $\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})$ with $J \cap \mathcal{I}(\mathfrak{q}) \neq \varnothing$. Then*

$$\mu(F \cap J) \leq 2^{200a^3 + 19} \operatorname{dens}_2(\mathfrak{T}(\mathfrak{u})) \mu(J) \,.$$

PROOF OF LEMMA 7.3.1. Denote

$$\mathcal{E}(\mathfrak{u}) = \bigcup_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{p}) \,.$$

Then we have

$$\left| \int_X \bar{g} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \, d\mu \right| = \left| \int_X \overline{g \mathbf{1}_{\mathcal{E}(\mathfrak{u})}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \, d\mu \right| \,.$$

By Lemma 7.2.1, this is bounded by

$$\leq 2^{104a^3} \|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|\|_2 \|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} |\mathbf{1}_{\mathcal{E}(\mathfrak{u})} g|\|_2 \,. \tag{7.3.4}$$

We bound the two factors separately. We have

$$\|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} |\mathbf{1}_{\mathcal{E}(\mathfrak{u})} g|\|_2 = \left(\sum_{L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))} \frac{1}{\mu(L)} \left(\int_{L \cap \mathcal{E}(\mathfrak{u})} |g(y)| \, d\mu(y) \right)^2 \right)^{1/2} \,.$$

By Cauchy-Schwarz and Lemma 7.3.2 this is at most

$$\leq \left(\sum_{L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))} 2^{101a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u})) \int_{L \cap \mathcal{E}(\mathfrak{u})} |g(y)|^2 \, d\mu(y) \right)^{1/2} \,.$$

Since cubes $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ are pairwise disjoint by Lemma 7.1.2, this is

$$\leq 2^{51a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2 \,. \tag{7.3.5}$$

Similarly, we have

$$\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2 = \left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \frac{1}{\mu(J)} \left(\int_J |f(y)| \, \mathrm{d}\mu(y) \right)^2 \right)^{1/2} . \qquad (7.3.6)$$

By Cauchy-Schwarz, this is

$$\leq \left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_J |f(y)|^2 \, \mathrm{d}\mu(y) \right)^{1/2} .$$

Since cubes in $\mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ are pairwise disjoint by Lemma 7.1.2, this at most

$$\|f\|_2 . \qquad (7.3.7)$$

Combining (7.3.4), (7.3.5) and (7.3.7) and using $a \geq 4$ gives (7.3.1).

If $f \leq \mathbf{1}_F$ then $f = f\mathbf{1}_F$, so

$$\left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_J |f(y)|^2 \, \mathrm{d}\mu(y) \right)^{1/2} = \left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_{J \cap F} |f(y)|^2 \, \mathrm{d}\mu(y) \right)^{1/2} .$$

We estimate as before, using now Lemma 7.3.3 and Cauchy-Schwarz, and obtain that this is

$$\leq 2^{100a^3+10} \, \mathrm{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \|f\|_2 .$$

Combining this with (7.3.4), (7.3.6) and $a \geq 4$ gives (7.3.2). $\qquad \square$

Now we prove the two auxiliary estimates.

PROOF OF LEMMA 7.3.2. If the set on the right hand side is empty, then (7.3.3) holds. If not, then there exists $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $L \cap \mathcal{I}(\mathfrak{p}) \neq \emptyset$.

Suppose first that there exists such $\mathfrak{p}$ with $\mathrm{s}(\mathfrak{p}) \leq s(L)$. Then by (2.0.8) $\mathcal{I}(\mathfrak{p}) \subset L$, which gives by the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ that $s(L) = -S$ and hence $L = \mathcal{I}(\mathfrak{p})$. Let $\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})$ with $E(\mathfrak{q}) \cap L \neq \emptyset$. Since $s(L) = -S \leq \mathrm{s}(\mathfrak{q})$ it follows from (2.0.8) that $\mathcal{I}(\mathfrak{p}) = L \subset \mathcal{I}(\mathfrak{q})$. We have then by Lemma 2.1.2

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{q})) \leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u})) + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\mathfrak{u}))$$
$$\leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u})) + d_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\mathfrak{u})) .$$

Using that $\mathfrak{p}, \mathfrak{q} \in \mathfrak{T}(\mathfrak{u})$ and (2.0.32), this is at most 8. Using again the triangle inequality and Lemma 2.1.2, we obtain that for each $q \in B_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), 1)$

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), q) \leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{q})) + d_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), q) \leq 9 .$$

Thus $L \cap E(\mathfrak{q}) \subset E_2(9, \mathfrak{p})$. We obtain

$$\mu(L \cap \bigcup_{\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{q})) \leq \mu(E_2(9, \mathfrak{p})) .$$

By the definition of dens_1, this is bounded by

$$9^a \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \mu(\mathcal{I}(\mathfrak{p})) = 9^a \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \mu(L) .$$

Since $a \geq 4$, (7.3.3) follows in this case.

Now suppose that for each $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $L \cap E(\mathfrak{p}) \neq \varnothing$, we have $s(\mathfrak{p}) > s(L)$. Since there exists at least one such $\mathfrak{p}$, there exists in particular at least one cube $L'' \in \mathcal{D}$ with $L \subset L''$ and $s(L'') > s(L)$. By (2.0.7), there exists $L' \in \mathcal{D}$ with $L \subset L'$ and $s(L') = s(L) + 1$. By the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ there exists a tile $\mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u})$ with $\mathcal{I}(\mathfrak{p}'') \subset L'$. Let $\mathfrak{p}'$ be the unique tile such that $\mathcal{I}(\mathfrak{p}') = L'$ and such that $\Omega(\mathfrak{u}) \cap \Omega(\mathfrak{p}') \neq \varnothing$. Since by (2.0.32) $s(\mathfrak{p}') = s(L') \leq s(\mathfrak{p}) < s(\mathfrak{u})$, we have by (2.0.8) and (2.0.14) that $\Omega(\mathfrak{u}) \subset \Omega(\mathfrak{p}')$. Let $\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})$ with $L \cap E(\mathfrak{q}) \neq \varnothing$. As shown above, this implies $s(\mathfrak{q}) \geq s(L')$, so by (2.0.8) $L' \subset \mathcal{I}(\mathfrak{q})$. If $q \in B_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), 1)$, then by a similar calculation as above, using the triangle inequality, Lemma 2.1.2 and (2.0.32), we obtain

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), q) \leq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{q})) + d_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), q) \leq 6 \,.$$

Thus $L \cap E(\mathfrak{q}) \subset E_2(6, \mathfrak{p}')$. Since $\mathcal{I}(\mathfrak{p}'') \subset \mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p})$ and $\mathfrak{p}'', \mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$, we have $\mathfrak{p}' \in \mathfrak{P}(\mathfrak{T}(\mathfrak{u}))$. We deduce using the definition (2.0.28) of dens_1

$$\mu\left(L \cap \bigcup_{\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{q})\right) \leq \mu(E_2(6, \mathfrak{q}')) \leq 6^a \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \mu(L') \,.$$

Using the doubling property (1.0.5), (2.0.10), and $a \geq 4$ this is estimated by

$$6^a 2^{100a^3 + 5} \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \mu(L) \leq 2^{101a^3} \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \mu(L) \,.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

PROOF OF LEMMA 7.3.3. Suppose first that there exists a tile $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1})$. By the definition of $\mathcal{J}(\mathfrak{T}(\mathfrak{u}))$, this implies that $s(J) = -S$, and in particular $s(\mathfrak{p}) \geq s(J)$. Using the triangle inequality and (2.0.10) it follows that $J \subset B(c(\mathfrak{p}), 200D^{s(\mathfrak{p})+1})$. From the doubling property (1.0.5), $D = 2^{100a^2}$ and (2.0.10), we obtain

$$\mu(\mathcal{I}(\mathfrak{p})) \leq 2^{100a^3 + 9} \mu(J)$$

and hence

$$\mu(B(c(\mathfrak{p}), 200D^{s(\mathfrak{p})+1})) \leq 2^{200a^3 + 19} \mu(J) \,.$$

With the definition (2.0.29) of dens_2 it follows that

$$\mu(J \cap F) \leq \mu(B(c(\mathfrak{p}), 200D^{s(\mathfrak{p})+1}) \cap F)$$

$$\leq \mathrm{dens}_2(\mathfrak{T}(\mathfrak{u})) \mu(B(c(\mathfrak{p}), 200D^{s(\mathfrak{p})+1})) \leq 2^{200a^3 + 19} \, \mathrm{dens}_2(\mathfrak{T}(\mathfrak{u})) \mu(J) \,,$$

completing the proof in this case.

Now suppose that there does not exist a tile $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1})$. If we had $s(\mathfrak{q}) \leq s(J)$, then by (2.0.8) and (2.0.10) $\mathcal{I}(\mathfrak{q}) \subset J \subset B(c(J), 100D^{s(J)+1})$, contradicting our assumption. Thus $s(\mathfrak{q}) > s(J)$. Then, by (2.0.7) and (2.0.8), there exists some cube $J' \in \mathcal{D}$ with $s(J') = s(J) + 1$ and $J \subset J'$. By definition of $\mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ there exists some $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ such that $\mathcal{I}(\mathfrak{p}) \subset B(c(J'), 100D^{s(J')+1})$. From the doubling property (1.0.5), $D = 2^{100a^2}$ and (2.0.10), we obtain

$$\mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})})) \leq 2^{4a} \mu(\mathcal{I}(\mathfrak{p})) \leq 2^{200a^3 + 14} \mu(J) \,. \qquad (7.3.8)$$

If $J \subset B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})})$, then we bound

$$\mu(J \cap F) \leq \mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})}) \cap F)$$

and use the definition (2.0.29)

$$\leq \operatorname{dens}_2(\mathfrak{T}(\mathfrak{u}))\mu(B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})})) \leq 2^{200a^3+14}\mu(J).$$

From now on we assume $J \not\subset B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})})$. Since

$$\mathrm{c}(\mathfrak{p}) \in \mathcal{J}(\mathfrak{p}) \subset B(c(J'), 100D^{\mathrm{s}(J')+1}),$$

we have by (2.0.10) and the triangle inequality

$$J \subset J' \subset B(c(J'), 4D^{\mathrm{s}(J')}) \subset B(\mathrm{c}(\mathfrak{p}), 104D^{\mathrm{s}(J')+1}).$$

In particular this implies $104D^{\mathrm{s}(J')+1} > 4D^{\mathrm{s}(\mathfrak{p})}$. By the triangle inequality we also have

$$B(\mathrm{c}(\mathfrak{p}), 104D^{\mathrm{s}(J')+1}) \subset B(c(J), 204D^{\mathrm{s}(J')+1}),$$

so from the doubling property (1.0.5)

$$\mu(B(\mathrm{c}(\mathfrak{p}), 104D^{\mathrm{s}(J')+1})) \leq 2^{200a^3+10}\mu(J).$$

From here one completes the proof as in the other cases. $\qquad\square$

7.4. Almost orthogonality of separated trees

The main result of this subsection is the almost orthogonality estimate for operators associated to distinct trees in a forest in Lemma 7.4.4 below. We will deduce it from Lemmas 7.4.5 and 7.4.6, which are proven in Subsections 7.5 and 7.6, respectively. Before stating it, we introduce some relevant notation.

The adjoint of the operator $T_\mathfrak{p}$ defined in (2.0.21) is given by

$$T_\mathfrak{p}^* g(x) = \int_{E(\mathfrak{p})} \overline{K_{\mathrm{s}(\mathfrak{p})}(y,x)} e(-Q(y)(x) + Q(y)(y)) g(y) \, \mathrm{d}\mu(y). \tag{7.4.1}$$

LEMMA 7.4.1 (adjoint tile support). *For each $\mathfrak{p} \in \mathfrak{P}$, we have*

$$T_\mathfrak{p}^* g = \mathbf{1}_{B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})})} T_\mathfrak{p}^* \mathbf{1}_{\mathcal{J}(\mathfrak{p})} g.$$

For each $\mathfrak{u} \in \mathfrak{U}$ and each $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$, we have

$$T_\mathfrak{p}^* g = \mathbf{1}_{\mathcal{J}(\mathfrak{u})} T_\mathfrak{p}^* \mathbf{1}_{\mathcal{J}(\mathfrak{u})} g.$$

PROOF. By (2.0.32), $E(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u})$. Thus by (7.4.1)

$$T_\mathfrak{p}^* g(x) = T_\mathfrak{p}^*(\mathbf{1}_{\mathcal{J}(\mathfrak{p})} g)(x)$$

$$= \int_{E(\mathfrak{p})} \overline{K_{\mathrm{s}(\mathfrak{p})}(y,x)} e(-Q(y)(x) + Q(y)(y)) \mathbf{1}_{\mathcal{J}(\mathfrak{p})}(y) g(y) \, \mathrm{d}\mu(y).$$

If this integral is not 0, then there exists $y \in \mathcal{J}(\mathfrak{p})$ such that $K_{\mathrm{s}(\mathfrak{p})}(y,x) \neq 0$. By (2.1.2), (2.0.10) and the triangle inequality, it follows that

$$x \in B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})}).$$

Thus

$$T_\mathfrak{p}^* g(x) = \mathbf{1}_{B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})})}(x) T_\mathfrak{p}^*(\mathbf{1}_{\mathcal{J}(\mathfrak{p})} g)(x).$$

The second claimed equation follows now since $\mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u})$ and by (2.0.37) $B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})}) \subset \mathcal{J}(\mathfrak{u})$. $\qquad\square$

LEMMA 7.4.2 (adjoint tree estimate). *For all bounded g with bounded support, we have that*

$$\left\| \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g \right\|_2 \leq 2^{155a^3} \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2 \,.$$

PROOF. By Cauchy-Schwarz and Lemma 7.3.1, we have for all bounded f, g with bounded support that

$$\left| \int_X \overline{\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g} f \,\mathrm{d}\mu \right| = \left| \int_X g \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \,\mathrm{d}\mu \right|$$

$$\leq 2^{155a^3} \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2 \|f\|_2 \,. \tag{7.4.2}$$

Let $f = \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g$. If g is bounded and has bounded support, then the same is true for f. In particular $\|f\|_2 < \infty$. Dividing (7.4.2) by $\|f\|_2$ completes the proof. □

We define

$$S_{2,\mathfrak{u}} g := \left| \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g \right| + M_{\mathcal{B},1} g + |g| \,.$$

LEMMA 7.4.3 (adjoint tree control). *We have for all bounded g with bounded support*

$$\|S_{2,\mathfrak{u}} g\|_2 \leq 2^{156a^3} \|g\|_2 \,.$$

PROOF. This follows immediately from Minkowski's inequality, Proposition 2.0.6 and Lemma 7.4.2, using that $a \geq 4$. □

Now we are ready to state the main result of this subsection.

LEMMA 7.4.4 (correlation separated trees). *For any $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ and all bounded g_1, g_2 with bounded support, we have*

$$\left| \int_X \sum_{\mathfrak{p}_1\in\mathfrak{T}(\mathfrak{u}_1)} \sum_{\mathfrak{p}_2\in\mathfrak{T}(\mathfrak{u}_2)} T_{\mathfrak{p}_1}^* g_1 \overline{T_{\mathfrak{p}_2}^* g_2} \,\mathrm{d}\mu \right| \tag{7.4.3}$$

$$\leq 2^{550a^3 - 3n} \prod_{j=1}^{2} \|S_{2,\mathfrak{u}_j} g_j\|_{L^2(\mathcal{J}(\mathfrak{u}_1)\cap\mathcal{J}(\mathfrak{u}_2))} \,. \tag{7.4.4}$$

PROOF OF LEMMA 7.4.4. By Lemma 7.4.1 and (2.0.8), the left hand side (7.4.3) is 0 unless $\mathcal{J}(\mathfrak{u}_1) \subset \mathcal{J}(\mathfrak{u}_2)$ or $\mathcal{J}(\mathfrak{u}_2) \subset \mathcal{J}(\mathfrak{u}_1)$. Without loss of generality we assume that $\mathcal{J}(\mathfrak{u}_1) \subset \mathcal{J}(\mathfrak{u}_2)$.

Define

$$\mathfrak{S} := \{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1) \cup \mathfrak{T}(\mathfrak{u}_2) \ : \ d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq 2^{Zn/2}\} \,. \tag{7.4.5}$$

Lemma 7.4.4 follows by combining the definition (2.0.3) of Z with the following two lemmas. □

LEMMA 7.4.5 (correlation distant tree parts). *We have for all $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ with $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$ and all bounded g_1, g_2 with bounded support*

$$\left| \int_X \sum_{\mathfrak{p}_1 \in \mathfrak{T}(\mathfrak{u}_1)} \sum_{\mathfrak{p}_2 \in \mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}} T_{\mathfrak{p}_1}^* g_1 \overline{T_{\mathfrak{p}_2}^* g_2} \, \mathrm{d}\mu \right| \tag{7.4.6}$$

$$\leq 2^{541 a^3} 2^{-Zn/(4a^2 + 2a^3)} \prod_{j=1}^2 \| S_{2,\mathfrak{u}_j} g_j \|_{L^2(\mathcal{I}(\mathfrak{u}_1))} \, . \tag{7.4.7}$$

LEMMA 7.4.6 (correlation near tree parts). *We have for all $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ with $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$ and all bounded g_1, g_2 with bounded support*

$$\left| \int_X \sum_{\mathfrak{p}_1 \in \mathfrak{T}(\mathfrak{u}_1)} \sum_{\mathfrak{p}_2 \in \mathfrak{T}(\mathfrak{u}_2) \setminus \mathfrak{S}} T_{\mathfrak{p}_1}^* g_1 \overline{T_{\mathfrak{p}_2}^* g_2} \, \mathrm{d}\mu \right| \tag{7.4.8}$$

$$\leq 2^{222 a^3} 2^{-Zn 2^{-10a}} \prod_{j=1}^2 \| S_{2,\mathfrak{u}_j} g_j \|_{L^2(\mathcal{I}(\mathfrak{u}_1))} \, . \tag{7.4.9}$$

In the proofs of both lemmas, we will need the following observation.

LEMMA 7.4.7 (overlap implies distance). *Let $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ with $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$. If $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1) \cup \mathfrak{T}(\mathfrak{u}_2)$ with $\mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{u}_1) \neq \varnothing$, then $\mathfrak{p} \in \mathfrak{S}$. In particular, we have $\mathfrak{T}(\mathfrak{u}_1) \subset \mathfrak{S}$.*

PROOF. Suppose first that $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1)$. Then $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$, by (2.0.32). Thus we have by the separation condition (2.0.36), (2.0.15), (2.0.32) and the triangle inequality

$$\begin{aligned} d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) &\geq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}_2)) - d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}_1)) \\ &\geq 2^{Z(n+1)} - 4 \\ &\geq 2^{Zn} \, , \end{aligned}$$

using that $Z = 2^{12a} \geq 4$. Hence $\mathfrak{p} \in \mathfrak{S}$.

Suppose now that $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2)$. If $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}_1)$, then the same argument as above with $\mathfrak{u}_1$ and $\mathfrak{u}_2$ swapped shows $\mathfrak{p} \in \mathfrak{S}$. If $\mathcal{I}(\mathfrak{p}) \not\subset \mathcal{I}(\mathfrak{u}_1)$ then, by (2.0.8), $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{p})$. Pick $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}_1)$, we have $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{p})$. Hence, by Lemma 2.1.2 and the first paragraph

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq 2^{Zn} \, ,$$

so $\mathfrak{p} \in \mathfrak{S}$. $\qquad \square$

To simplify the notation, we will write at various places throughout the proof of Lemmas 7.4.5 and 7.4.6 for a subset $\mathfrak{C} \subset \mathfrak{P}$

$$T_{\mathfrak{C}} f := \sum_{\mathfrak{p} \in \mathfrak{C}} T_{\mathfrak{p}} f \, , \qquad T_{\mathfrak{C}}^* g := \sum_{\mathfrak{p} \in \mathfrak{C}} T_{\mathfrak{p}}^* g \, .$$

7.5. Proof of the Tiles with large separation Lemma

Lemma 7.4.5 follows from the van der Corput estimate in Proposition 2.0.5. We apply this proposition in Section 7.5.3. To prepare this application, we first, in Section 7.5.1, construct a suitable partition of unity, and show then, in Section 7.5.2 the Hölder estimates needed to apply Proposition 2.0.5.

7.5.1. A partition of unity. Define

$$\mathcal{J}' = \{J \in \mathcal{J}(\mathfrak{S}) \ : \ J \subset \mathcal{I}(\mathfrak{u}_1)\} \,.$$

LEMMA 7.5.1 (dyadic partition 1). *We have that*

$$\mathcal{I}(\mathfrak{u}_1) = \dot{\bigcup_{J \in \mathcal{J}'}} J \,.$$

PROOF. By Lemma 7.1.2, it remains only to show that each $J \in \mathcal{J}(\mathfrak{S})$ with $J \cap \mathcal{I}(\mathfrak{u}_1) \neq \varnothing$ is in $\mathcal{J}'$. But if $J \notin \mathcal{J}'$, then by (2.0.8) $\mathcal{I}(\mathfrak{u}_1) \subsetneq J$. Pick $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1) \subset \mathfrak{S}$. Then $\mathcal{I}(\mathfrak{p}) \subsetneq J$. This contradicts the definition of $\mathcal{J}(\mathfrak{S})$. $\qquad\square$

For cubes $J \in \mathcal{D}$, denote

$$B(J) := B(c(J), 8D^{s(J)}). \tag{7.5.1}$$

The main result of this subsubsection is the following.

LEMMA 7.5.2 (Lipschitz partition unity). *There exists a family of functions χ_J, $J \in \mathcal{J}'$ such that*

$$\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} = \sum_{J \in \mathcal{J}'} \chi_J \,, \tag{7.5.2}$$

and for all $J \in \mathcal{J}'$ and all $y, y' \in \mathcal{I}(\mathfrak{u}_1)$

$$0 \leq \chi_J(y) \leq \mathbf{1}_{B(J)}(y) \,, \tag{7.5.3}$$

$$|\chi_J(y) - \chi_J(y')| \leq 2^{226a^3} \frac{\rho(y, y')}{D^{s(J)}} \,. \tag{7.5.4}$$

In the proof, we will use the following auxiliary lemma.

LEMMA 7.5.3 (moderate scale change). *If $J, J' \in \mathcal{J}'$ with*

$$B(J) \cap B(J') \neq \varnothing \,,$$

then $|s(J) - s(J')| \leq 1$.

PROOF OF LEMMA 7.5.2. For each cube $J \in \mathcal{J}$ let

$$\tilde{\chi}_J(y) = \max\{0, 8 - D^{-s(J)}\rho(y, c(J))\} \,,$$

and set

$$a(y) = \sum_{J \in \mathcal{J}'} \tilde{\chi}_J(y) \,.$$

We define

$$\chi_J(y) := \frac{\tilde{\chi}_J(y)}{a(y)} \,.$$

Then, due to (2.0.37) and (7.5.1), the properties (7.5.2) and (7.5.3) are clearly true. Estimate (7.5.4) follows from (7.5.3) if $y, y' \notin B(J)$. Thus we can assume that $y \in B(J)$. We have by the triangle inequality

$$|\chi_J(y) - \chi_J(y')| \leq \frac{|\tilde{\chi}_J(y) - \tilde{\chi}_J(y')|}{a(y)} + \frac{\tilde{\chi}_J(y')|a(y) - a(y')|}{a(y)a(y')}$$

Since $\tilde{\chi}_J(z) \geq 4$ for all $z \in B(c(J), 4) \supset J$ and by Lemma 7.5.1, we have that $a(z) \geq 4$ for all $z \in \mathcal{J}(\mathfrak{u}_1)$. So we can estimate the above further by

$$\leq 2^{-2}(|\tilde{\chi}_J(y) - \tilde{\chi}_J(y')| + \tilde{\chi}_J(y')|a(y) - a(y')|).$$

If $y' \notin B(c(\mathfrak{p}), 8D^{s(\mathfrak{p})})$ then the second summand vanishes. Else, we can estimate the above, using also that $|\tilde{\chi}_J(y')| \leq 8$, by

$$\leq 2^{-2}|\tilde{\chi}_J(y) - \tilde{\chi}_J(y')| + 2 \sum_{\substack{J' \in \mathcal{J}' \\ B(c(J'), 8D^{s(J')}) \cap B(c(J), 8D^{s(J)}) \neq \varnothing}} |\tilde{\chi}_{J'}(y) - \tilde{\chi}_{J'}(y')|.$$

By the triangle inequality, we have for all dyadic cubes $I \in \mathcal{J}'$

$$|\tilde{\chi}_I(y) - \tilde{\chi}_I(y')| \leq \rho(y, y')D^{-s(I)}.$$

Using this above, we obtain

$$|\chi_J(y) - \chi_J(y')| \leq \rho(y, y')\Big(\frac{1}{4}D^{-s(J)} + 2 \sum_{\substack{J' \in \mathcal{J}' \\ B(J') \cap B(J) \neq \varnothing}} D^{-s(J')}\Big).$$

By Lemma 7.5.3, this is at most

$$\frac{\rho(y, y')}{D^{s(J)}} \left(\frac{1}{4} + 2D|\{J' \in \mathcal{J}' \; : \; B(J') \cap B(J) \neq \varnothing\}|\right).$$

By (2.0.10) and Lemma 7.5.1, the balls $B(c(J'), \frac{1}{4}D^{s(J')})$ are pairwise disjoint, so by Lemma 7.5.3 the balls $B(c(J'), \frac{1}{4}D^{s(J)-1})$ are also disjoint. By the triangle inequality and Lemma 7.5.3, each such ball for J' in the set of the last display is contained in

$$B(c(J), 9D^{s(J)+1}).$$

By the doubling property (1.0.5), we further have

$$\mu\Big(B(c(J'), \frac{1}{4}D^{s(J')})\Big) \geq 2^{-200a^3-6}\mu(B(c(J), 9D^{s(J)+1}))$$

for each such ball. Thus

$$|\{J' \in \mathcal{J}' \; : \; B(J') \cap B(J) \neq \varnothing\}| \leq 2^{200a^3+6}.$$

Recalling that $D = 2^{100a^2}$, we obtain

$$\frac{1}{4} + 2D|\{J' \in \mathcal{J}' \; : \; B(J') \cap B(J) \neq \varnothing\}| \leq 2^{200a^3+100a^2+8}.$$

Since $a \geq 4$, (7.5.4) follows. $\qquad\square$

PROOF OF LEMMA 7.5.3. Suppose that $s(J') < s(J) - 1$. Then $s(J) > -S$. Thus, by the definition of $\mathcal{J}'$ there exists no $\mathfrak{p} \in \mathfrak{S}$ with

$$\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1}) . \tag{7.5.5}$$

Since $s(J') < s(J)$ and $J', J \subset \mathcal{I}(\mathfrak{u}_1)$, we have $J' \subsetneq \mathcal{I}(\mathfrak{u}_1)$. By (2.0.7), (2.0.8) there exists a cube $J'' \in \mathcal{D}$ with $J \subset J''$ and $s(J'') = s(J') + 1$. By the definition of $\mathcal{J}'$, there exists a tile $\mathfrak{p} \in \mathfrak{S}$ with

$$\mathcal{I}(\mathfrak{p}) \subset B(c(J''), 100D^{s(J')+2}) . \tag{7.5.6}$$

But by the triangle inequality and (2.0.1), we have

$$B(c(J''), 100D^{s(J')+2}) \subset B(c(J), 100D^{s(J)+1}) ,$$

which contradicts (7.5.5) and (7.5.6). $\qquad\square$

7.5.2. Hölder estimates for adjoint tree operators. Let $g_1, g_2 : X \to \mathbb{C}$ be bounded with bounded support. Define for $J \in \mathcal{J}'$

$$h_J(y) := \chi_J(y) \cdot (e(\mathcal{Q}(\mathfrak{u}_1)(y))T^*_{\mathfrak{T}(\mathfrak{u}_1)}g_1(y)) \cdot \overline{(e(\mathcal{Q}(\mathfrak{u}_2)(y))T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2(y))} . \tag{7.5.7}$$

The main result of this subsubsection is the following τ-Hölder estimate for h_J, where $\tau = 1/a$.

LEMMA 7.5.4 (Holder correlation tree). *We have for all $J \in \mathcal{J}'$ that*

$$\|h_J\|_{C^\tau(B(c(J), 8D^{s(J)}))} \le 2^{535a^3} \prod_{j=1,2} \Big(\inf_{B(c(J), \frac{1}{8}D^{s(J)})} |T^*_{\mathfrak{T}(\mathfrak{u}_j)}g_j| + \inf_J M_{\mathcal{B},1}|g_j| \Big) . \tag{7.5.8}$$

We will prove this lemma at the end of this section, after establishing several auxiliary results.

We begin with the following Hölder continuity estimate for adjoints of operators associated to tiles.

LEMMA 7.5.5 (Holder correlation tile). *Let $\mathfrak{u} \in \mathfrak{U}$ and $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$. Then for all $y, y' \in X$ and all bounded g with bounded support, we have*

$$|e(\mathcal{Q}(\mathfrak{u})(y))T^*_{\mathfrak{p}}g(y) - e(\mathcal{Q}(\mathfrak{u})(y'))T^*_{\mathfrak{p}}g(y')|$$

$$\le \frac{2^{151a^3}}{\mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})}))} \left(\frac{\rho(y, y')}{D^{s(\mathfrak{p})}} \right)^{1/a} \int_{E(\mathfrak{p})} |g(x)| \, d\mu(x) . \tag{7.5.9}$$

PROOF. By (7.4.1), we have

$$|e(\mathcal{Q}(\mathfrak{u})(y))T^*_{\mathfrak{p}}g(y) - e(\mathcal{Q}(\mathfrak{u})(y'))T^*_{\mathfrak{p}}g(y')|$$

$$= \left| \int_{E(\mathfrak{p})} e(Q(x)(x) - Q(x)(y) + \mathcal{Q}(\mathfrak{u})(y))\overline{K_{s(\mathfrak{p})}(x, y)}g(x) \right.$$

$$\left. - e(Q(x)(x) - Q(x)(y') + \mathcal{Q}(\mathfrak{u})(y'))\overline{K_{s(\mathfrak{p})}(x, y')}g(x) \, d\mu(x) \right|$$

$$\leq \int_{E(\mathfrak{p})} |g(x)||e(Q(x)(y) - Q(x)(y') - \mathcal{Q}(\mathfrak{u})(y) + \mathcal{Q}(\mathfrak{u})(y'))\overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)}$$

$$- \overline{K_{\mathrm{s}(\mathfrak{p})}(x,y')}|\,\mathrm{d}\mu(x)$$

$$\leq \int_{E(\mathfrak{p})} |g(x)||e(-Q(x)(y) + Q(x)(y') + \mathcal{Q}(\mathfrak{u})(y) - \mathcal{Q}(\mathfrak{u})(y')) - 1|$$

$$\times |\overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)}|\,\mathrm{d}\mu(x) \quad (7.5.10)$$

$$+ \int_{E(\mathfrak{p})} |g(x)||\overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)} - \overline{K_{\mathrm{s}(\mathfrak{p})}(x,y')}|\,\mathrm{d}\mu(x). \quad (7.5.11)$$

By the oscillation estimate (1.0.7), we have

$$|-Q(x)(y) + Q(x)(y') + \mathcal{Q}(\mathfrak{u})(y) - \mathcal{Q}(\mathfrak{u})(y')|$$

$$\leq d_{B(y,\rho(y,y'))}(Q(x), \mathcal{Q}(\mathfrak{u})). \quad (7.5.12)$$

Suppose that $y, y' \in B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})})$, so that $\rho(y,y') \leq 10D^{\mathrm{s}(\mathfrak{p})}$. Let $k \in \mathbb{Z}$ be such that $2^{ak}\rho(y,y') \leq 10D^{\mathrm{s}(\mathfrak{p})}$ but $2^{a(k+1)}\rho(y,y') > 10D^{\mathrm{s}(\mathfrak{p})}$. In particular, $k \geq 0$. Then, using (1.0.8), we can bound (7.5.12) from above by

$$2^{-k}d_{B(\mathrm{c}(\mathfrak{p}),10D^{\mathrm{s}(\mathfrak{p})})}(Q(x), \mathcal{Q}(\mathfrak{u})) \leq 2^{6a-k}d_{\mathfrak{p}}(Q(x), \mathcal{Q}(\mathfrak{u})).$$

Since $x \in E(\mathfrak{p})$ we have $Q(x) \in \Omega(\mathfrak{p}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}),1)$, and since $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ we have $\mathcal{Q}(\mathfrak{u}) \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}),4)$, so this is estimated by

$$\leq 5 \cdot 2^{6a-k}.$$

By definition of k, we have

$$k \leq \frac{1}{a}\log_2\left(\frac{10D^{\mathrm{s}(\mathfrak{p})}}{\rho(y,y')}\right),$$

which gives

$$|-Q(x)(y) + Q(x)(y') + \mathcal{Q}(\mathfrak{u})(y) - \mathcal{Q}(\mathfrak{u})(y')| \leq 5 \cdot 2^{6a}\left(\frac{\rho(y,y')}{10D^{\mathrm{s}(\mathfrak{p})}}\right)^{1/a}. \quad (7.5.13)$$

For all $x \in \mathcal{I}(\mathfrak{p})$, we have by (1.0.5) that

$$\mu(B(x, D^{\mathrm{s}(\mathfrak{p})})) \geq 2^{-3a}\mu(B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})})).$$

Combining the above with (2.1.3), (2.1.4) and (7.5.13), we obtain

$$(7.5.10) + (7.5.11) \leq \frac{2^{3a}}{\mu(B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})}))}\int_{E(\mathfrak{p})} |g(x)|\,\mathrm{d}\mu(x) \times$$

$$(2^{102a^3} \cdot 5 \cdot 2^{6a}\left(\frac{\rho(y,y')}{D^{\mathrm{s}(\mathfrak{p})}}\right)^{1/a} + 2^{150a^3}\left(\frac{\rho(y,y')}{D^{\mathrm{s}(\mathfrak{p})}}\right)^{1/a})$$

Since $\rho(y,y') \leq 10D^{\mathrm{s}(\mathfrak{p})}$, we conclude

$$(7.5.10) + (7.5.11) \leq \frac{2^{151a^3}}{\mu(B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})}))}\left(\frac{\rho(y,y')}{D^{\mathrm{s}(\mathfrak{p})}}\right)^{1/a}\int_{E(\mathfrak{p})} |g(x)|\,\mathrm{d}\mu(x).$$

Next, if $y, y' \notin B(c(\mathfrak{p}), 5D^{s(\mathfrak{p})})$, then $T_{\mathfrak{p}}^* g(y) = T_{\mathfrak{p}}^* g(y') = 0$, by Lemma 7.4.1. Then (7.5.9) holds.

Finally, if $y \in B(c(\mathfrak{p}), 5D^{s(\mathfrak{p})})$ and $y' \notin B(c(\mathfrak{p}), 5D^{s(\mathfrak{p})})$, then

$$|e(Q(\mathfrak{u})(y))T_{\mathfrak{p}}^* g(y) - e(Q(\mathfrak{u})(y'))T_{\mathfrak{p}}^* g(y')| = |T_{\mathfrak{p}}^* g(y)|$$

$$\leq \int_{E(\mathfrak{p})} |K_{s(\mathfrak{p})}(x, y)||g(x)| \, \mathrm{d}\mu(x) \,.$$

By the same argument used to prove (2.1.6), this is bounded by

$$\leq 2^{102a^3} \int_{E(\mathfrak{p})} \frac{1}{\mu(B(x, D^s))} \psi(D^{-s}\rho(x, y))|g(x)| \, \mathrm{d}\mu(x) \,. \qquad (7.5.14)$$

It follows from the definition of ψ that

$$\psi(x) \leq \max\{0, (2 - 4x)^{1/a}\} \,.$$

Now for all $x \in E(\mathfrak{p})$, it follows by the triangle inequality and (2.0.10) that

$$2 - 4D^{-s(\mathfrak{p})}\rho(x, y) \leq 2 - 4D^{-s(\mathfrak{p})}\rho(y, c(\mathfrak{p})) + 4D^{-s(\mathfrak{p})}\rho(x, c(\mathfrak{p}))$$

$$\leq 18 - 4D^{-s(\mathfrak{p})}\rho(y, c(\mathfrak{p})) \leq 4D^{-s(\mathfrak{p})}\rho(y, y') - 2 \,.$$

Combining the above with the previous estimate on ψ, we get

$$\psi(D^{-s(\mathfrak{p})}\rho(x, y)) \leq 4(D^{-s(\mathfrak{p})}\rho(y, y'))^{1/a} \,.$$

Further, we obtain from the doubling property (1.0.5) and (2.0.10) that

$$\mu(B(x, D^s)) \geq 2^{-2a}\mu(c(\mathfrak{p}), 4D^s) \,.$$

Plugging this into (7.5.14) and using $a \geq 4$, we get

$$|T_{\mathfrak{p}}^* g(y)| \leq \frac{2^{103a^3}}{\mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})}))} \left(\frac{\rho(y, y')}{D^{s(\mathfrak{p})}}\right)^{1/a} \int_{E(\mathfrak{p})} |g(x)| \, \mathrm{d}\mu(y) \,,$$

which completes the proof of the lemma. $\qquad \square$

Recall that

$$B(J) := B(c(J), 8D^{s(J)}) \,.$$

We also denote

$$B^{\circ}(J) := B(c(J), \frac{1}{8}D^{s(J)}) \,.$$

LEMMA 7.5.6 (limited scale impact). *Let $\mathfrak{p} \in \mathfrak{T}_2 \smallsetminus \mathfrak{S}$, $J \in \mathcal{J}'$ and suppose that*

$$B(\mathcal{J}(\mathfrak{p})) \cap B^{\circ}(J) \neq \varnothing \,.$$

Then

$$s(J) \leq s(\mathfrak{p}) \leq s(J) + 3 \,.$$

PROOF. For the first estimate, assume that $s(\mathfrak{p}) < s(J)$, then in particular $s(\mathfrak{p}) \leq s(\mathfrak{u}_1)$. Since $\mathfrak{p} \notin \mathfrak{S}$, we have by Lemma 7.4.7 that $\mathcal{J}(\mathfrak{p}) \cap \mathcal{J}(\mathfrak{u}_1) = \varnothing$. Since $B\left(c(J), \frac{1}{4}D^{s(J)}\right) \subset \mathcal{J}(J) \subset \mathcal{J}(\mathfrak{u}_1)$, this implies

$$\rho(c(J), c(\mathfrak{p})) \geq \frac{1}{4}D^{s(J)} \,.$$

On the other hand

$$\rho(c(J), c(\mathfrak{p})) \leq \frac{1}{8}D^{s(J)} + 8D^{s(\mathfrak{p})},$$

by our assumption. Thus $D^{s(\mathfrak{p})} \geq 64D^{s(J)}$, which contradicts (2.0.1) and $a \geq 4$.

For the second estimate, assume that $s(\mathfrak{p}) > s(J) + 3$. Since $J \in \mathcal{J}'$, we have $J \subsetneq \mathcal{I}(\mathfrak{u}_1)$. Thus there exists $J' \in \mathcal{D}$ with $J \subset J'$ and $s(J') = s(J) + 1$, by (2.0.7) and (2.0.8). By definition of $\mathcal{J}'$, there exists some $\mathfrak{p}' \in \mathfrak{S}$ such that $\mathcal{I}(\mathfrak{p}') \subset B(c(J'), 100D^{s(J)+2})$. On the other hand, since $B(\mathcal{I}(\mathfrak{p})) \cap B^{\circ}(J) \neq \varnothing$, by the triangle inequality it holds that

$$B(c(J'), 100D^{s(J)+3}) \subset B(c(\mathfrak{p}), 10D^{s(\mathfrak{p})}).$$

Using the definition of $\mathfrak{S}$, we have

$$2^{Zn/2} \leq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \leq d_{B(c(J'), 100D^{s(J)+2})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)).$$

By (1.0.10), this is

$$\leq 2^{-100a}d_{B(c(J'), 100D^{s(J)+3})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2))$$

$$\leq 2^{-100a}d_{B(c(\mathfrak{p}), 10D^{s(\mathfrak{p})})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)),$$

and by (1.0.8) and the definition of $\mathfrak{S}$

$$\leq 2^{-94a}d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \leq 2^{-94a}2^{Zn/2}.$$

This is a contradiction, the second estimate follows. $\qquad\square$

LEMMA 7.5.7 (local tree control). *For all $J \in \mathcal{J}'$ and all bounded g with bounded support*

$$\sup_{B^{\circ}(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}} g| \leq 2^{104a^3} \inf_J M_{\mathcal{B},1}|g|$$

PROOF. By the triangle inequality and since $T^*_{\mathfrak{p}} g = \mathbf{1}_{B(c(\mathfrak{p}), 5D^{s(\mathfrak{p})})} T^*_{\mathfrak{p}} g$, we have

$$\sup_{B^{\circ}(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}} g| \leq \sup_{B^{\circ}(J)} \sum_{\substack{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S} \\ B(\mathcal{I}(\mathfrak{p})) \cap B^{\circ}(J) \neq \varnothing}} |T^*_{\mathfrak{p}} g|.$$

By Lemma 7.5.6, this is at most

$$\sum_{s=s(J)}^{s(J)+3} \sum_{\substack{\mathfrak{p} \in \mathfrak{P}, s(\mathfrak{p})=s \\ B(\mathcal{I}(\mathfrak{p})) \cap B^{\circ}(J) \neq \varnothing}} \sup_{B^{\circ}(J)} |T^*_{\mathfrak{p}} g|. \tag{7.5.15}$$

If $x \in E(\mathfrak{p})$ and $B(\mathcal{I}(\mathfrak{p})) \cap B^{\circ}(J) \neq \varnothing$, then

$$B(c(J), 16D^{s(\mathfrak{p})}) \subset B(x, 32D^{s(\mathfrak{p})}),$$

by (2.0.10) and the triangle inequality. Using the doubling property (1.0.5), it follows that

$$\mu(B(x, D^{s(\mathfrak{p})})) \geq 2^{-5a}\mu(B(c(J), 16D^{s(\mathfrak{p})})).$$

Using (7.4.1), (2.1.3) and that $a \geq 4$, we bound (7.5.15) by

$$2^{103a^3} \sum_{s=s(J)}^{s(J)+3} \sum_{\substack{\mathfrak{p} \in \mathfrak{P}, s(\mathfrak{p})=s \\ B(\mathcal{I}(\mathfrak{p})) \cap B^\circ(J) \neq \varnothing}} \frac{1}{\mu(B(c(J), 16D^s))} \int_{E(\mathfrak{p})} |g| \, \mathrm{d}\mu \,.$$

For each $I \in \mathcal{D}$, the sets $E(\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{P}$ with $\mathcal{I}(\mathfrak{p}) = I$ are pairwise disjoint by (2.0.20) and (2.0.13). Further, if $B(\mathcal{I}(\mathfrak{p})) \cap B^\circ(J) \neq \varnothing$ and $s(\mathfrak{p}) \geq s(J)$, then $E(\mathfrak{p}) \subset B(c(J), 32D^{s(\mathfrak{p})})$. Thus the last display is bounded by

$$2^{103a^3} \sum_{s=s(J)}^{s(J)+3} \frac{1}{\mu(B(c(J), 32D^s))} \int_{B(c(J), 16D^s)} |g| \, \mathrm{d}\mu \,.$$

$$\leq \inf_{x' \in J} 2^{103a^3+2} M_{\mathcal{B},1} |g| \,.$$

The lemma follows since $a \geq 4$. $\qquad \square$

LEMMA 7.5.8 (scales impacting interval). *Let* $\mathfrak{C} = \mathfrak{T}(\mathfrak{u}_1)$ *or* $\mathfrak{C} = \mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}$. *Then for each* $J \in \mathcal{J}'$ *and* $\mathfrak{p} \in \mathfrak{C}$ *with* $B(\mathcal{I}(\mathfrak{p})) \cap B(J) \neq \varnothing$, *we have* $s(\mathfrak{p}) \geq s(J)$.

PROOF. By Lemma 7.4.7, we have that in both cases, $\mathfrak{C} \subset \mathfrak{S}$. If $\mathfrak{p} \in \mathfrak{C}$ with $B(\mathcal{I}(\mathfrak{p})) \cap B(J) \neq \varnothing$ and $s(\mathfrak{p}) < s(J)$, then $\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1})$. Since $\mathfrak{p} \in \mathfrak{S}$, it follows from the definition of $\mathcal{J}'$ that $s(J) = -S$, which contradicts $s(\mathfrak{p}) < s(J)$. $\qquad \square$

LEMMA 7.5.9 (global tree control 1). *Let* $\mathfrak{C} = \mathfrak{T}(\mathfrak{u}_1)$ *or* $\mathfrak{C} = \mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}$. *Then for each* $J \in \mathcal{J}'$ *and all bounded* g *with bounded support, we have*

$$\sup_{B(J)} |T_{\mathfrak{C}}^* g| \leq \inf_{B^\circ(J)} |T_{\mathfrak{C}}^* g| + 2^{154a^3} \inf_J M_{\mathcal{B},1} |g| \qquad (7.5.16)$$

and for all $y, y' \in B(J)$

$$|e(\mathcal{Q}(\mathfrak{u})(y)) T_{\mathfrak{C}}^* g(y) - e(\mathcal{Q}(\mathfrak{u})(y')) T_{\mathfrak{C}}^* g(y')|$$

$$\leq 2^{153a^3} \left(\frac{\rho(y, y')}{D^{s(J)}} \right)^{1/a} \inf_J M_{\mathcal{B},1} |g| \,. \qquad (7.5.17)$$

PROOF. Note that (7.5.16) follows from (7.5.17), since for $y' \in B^\circ(J)$, by the triangle inequality,

$$\left(\frac{\rho(y, y')}{D^{s(J)}} \right)^{1/a} \leq \left(8 + \frac{1}{8} \right)^{1/a} \leq 2^{a^3} \,.$$

By the triangle inequality, Lemma 7.4.1 and Lemma 7.5.5, we have for all $y, y' \in B(J)$

$$|e(\mathcal{Q}(\mathfrak{u})(y)) T_{\mathfrak{C}}^* g(y) - e(\mathcal{Q}(\mathfrak{u})(y')) T_{\mathfrak{C}}^* g(y')| \qquad (7.5.18)$$

$$\leq \sum_{\substack{\mathfrak{p} \in \mathfrak{C} \\ B(\mathcal{I}(\mathfrak{p})) \cap B(J) \neq \varnothing}} |e(\mathcal{Q}(\mathfrak{u})(y)) T_{\mathfrak{p}}^* g(y) - e(\mathcal{Q}(\mathfrak{u})(y')) T_{\mathfrak{p}}^* g(y')|$$

$$\leq 2^{151a^3} \rho(y, y')^{1/a} \sum_{\substack{\mathfrak{p} \in \mathfrak{C} \\ B(\mathcal{I}(\mathfrak{p})) \cap B(J) \neq \varnothing}} \frac{D^{-s(\mathfrak{p})/a}}{\mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})}))} \int_{E(\mathfrak{p})} |g| \, \mathrm{d}\mu \,.$$

By Lemma 7.5.8, we have $s(\mathfrak{p}) \geq s(J)$ for all $\mathfrak{p}$ occurring in the sum. Further, for each $s \geq s(J)$, the sets $E(\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{P}$ with $s(\mathfrak{p}) = s$ are pairwise disjoint by (2.0.20) and (2.0.13), and contained in $B(c(J), 32D^s)$ by (2.0.10) and the triangle inequality. Using also the doubling estimate (1.0.5), we obtain that the expression in the last display can be estimated by

$$2^{151a^3} \rho(y, y')^{1/a} \sum_{S \geq s \geq s(J)} D^{-s/a} \frac{2^{3a}}{\mu(B(c(J), 32D^s))} \int_{B(c(J), 32D^s)} |g| \, \mathrm{d}\mu$$

$$\leq 2^{152a^3} \left(\frac{\rho(y, y')}{D^s} \right)^{1/a} \sum_{S \geq s \geq s(J)} D^{(s(J)-s)/a} \inf_J M_{\mathcal{B},1} |g| \, .$$

By convexity of $t \mapsto D^t$ and since $D \geq 2$, we have for all $-1 \leq t \leq 0$

$$D^t \leq 1 + t(1 - \frac{1}{D}) \leq 1 + \frac{1}{2}t \, .$$

Since $-1 \leq -1/a < 0$, it follows that

$$\sum_{S \geq s \geq s(J)} D^{(s(J)-s)/a} \leq \frac{1}{1 - D^{-1/a}} \leq 2a \leq 2^a \, .$$

Estimate (7.5.17), and therefore the lemma, follow. $\qquad\square$

LEMMA 7.5.10 (global tree control 2). *We have for all $J \in \mathcal{J}'$ and all bounded g with bounded support*

$$\sup_{B(J)} |T^*_{\mathfrak{T}_2 \cap \mathfrak{S}} g| \leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}_2} g| + 2^{155a^3} \inf_J M_{\mathcal{B},1} |g| \, .$$

PROOF. By Lemma 7.5.9

$$\sup_{B(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}} g| \leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}} g| + 2^{154a^3} \inf_J M_{\mathcal{B},1} |g|$$

$$\leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)} g| + \sup_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}} g| + 2^{154a^3} \inf_J M_{\mathcal{B},1} |g| \, ,$$

and by Lemma 7.5.7

$$\leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)} g| + (2^{104a^3} + 2^{154a^3}) \inf_J M_{\mathcal{B},1} |g| \, .$$

This completes the proof. $\qquad\square$

PROOF OF LEMMA 7.5.4. Let P be the product on the right hand side of (7.5.8), and h_J as defined in (7.5.7).

By (7.5.3) and Lemma 7.4.1, the function h_J is supported in $B(J) \cap \mathcal{J}(\mathfrak{u}_1)$. By (7.5.3) and Lemma 7.5.9, we have for all $y \in B(J)$:

$$|h_J(y)| \leq 2^{308a^3} P \, .$$

We have by the triangle inequality

$$|h_J(y) - h_J(y')|$$

$$\leq |\chi_J(y) - \chi_J(y')||T^*_{\mathfrak{T}(\mathfrak{u}_1)}g_1(y)||T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2(y)| \tag{7.5.19}$$

$$+ |\chi_J(y')||T^*_{\mathfrak{T}(\mathfrak{u}_1)}g_1(y) - T^*_{\mathfrak{T}(\mathfrak{u}_1)}g_1(y')||T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2(y)| \tag{7.5.20}$$

$$+ |\chi_J(y')||T^*_{\mathfrak{T}(\mathfrak{u}_1)}g_1(y')||T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2(y) - T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2(y')|\,. \tag{7.5.21}$$

As h_J is supported in $\mathcal{J}(\mathfrak{u}_1)$, we can assume without loss of generality that $y' \in \mathcal{J}(\mathfrak{u}_1)$. If $y \notin \mathcal{J}(\mathfrak{u}_1)$, then (7.5.19) vanishes. If $y \in \mathcal{J}(\mathfrak{u}_1)$ then we have by (7.5.4), Lemma 7.5.9 and Lemma 7.5.10

$$(7.5.19) \leq 2^{534a^3}\frac{\rho(y,y')}{D^{s(J)}}P\,,$$

where P denotes the product on the right hand side of (7.5.8).

By (7.5.3), Lemma 7.5.9 and Lemma 7.5.10, we have

$$(7.5.20) \leq 2^{310a^3}P\,.$$

By (7.5.3), and twice Lemma 7.5.9, we have

$$(7.5.21) \leq 2^{308a^3}P\,.$$

Using that $\rho(y,y') \leq 16D^{s(J)}$ and $a \geq 4$, the lemma follows. $\qquad\square$

7.5.3. The van der Corput estimate.

LEMMA 7.5.11 (lower oscillation bound). *For all $J \in \mathcal{J}'$, we have that*

$$d_{B(J)}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq 2^{-201a^3}2^{Zn/2}\,.$$

PROOF. Since $\varnothing \neq \mathfrak{T}(\mathfrak{u}_1) \subset \mathfrak{S}$ by Lemma 7.4.7, there exists at least one tile $\mathfrak{p} \in \mathcal{S}$ with $\mathcal{J}(\mathfrak{p}) \subsetneq \mathcal{J}(\mathfrak{u}_1)$. Thus $\mathcal{J}(\mathfrak{u}_1) \notin \mathcal{J}'$, so $J \subsetneq \mathcal{J}(\mathfrak{u}_1)$. Thus there exists a cube $J' \in \mathcal{D}$ with $J \subset J'$ and $s(J') = s(J) + 1$, by (2.0.7) and (2.0.8). By definition of $\mathcal{J}'$ and the triangle inequality, there exists $\mathfrak{p} \in \mathfrak{S}$ such that

$$\mathcal{J}(\mathfrak{p}) \subset B(c(J'), 100D^{s(J')+1}) \subset B(c(J), 128D^{s(J)+2})\,.$$

Thus, by definition of $\mathfrak{S}$:

$$2^{Zn/2} \leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \leq d_{B(c(J),128D^{s(J)+2})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2))\,.$$

By the doubling property (1.0.8), this is

$$\leq 2^{200a^3+4a}d_{B(J)}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2))\,,$$

which gives the lemma using $a \geq 4$. $\qquad\square$

Now we are ready to prove Lemma 7.4.5.

PROOF OF LEMMA 7.4.5. We have

$$(7.4.6) = \left|\int_X T^*_{\mathfrak{T}(\mathfrak{u}_1)}g_1\overline{T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2}\right|\,.$$

By Lemma 7.4.1, the right hand side is supported in $\mathcal{J}(\mathfrak{u}_1)$. Using (7.5.2) of Lemma 7.5.2 and the definition (7.5.7) of h_J, we thus have

$$\leq \sum_{J \in \mathcal{J}'} \left| \int_{B(J)} e(\mathcal{Q}(\mathfrak{u}_2)(y) - \mathcal{Q}(\mathfrak{u}_1)(y)) h_J(y) \, \mathrm{d}\mu(y) \right| .$$

Using Proposition 2.0.5 with the ball $B(J)$, we bound this by

$$\leq 2^{8a} \sum_{J \in \mathcal{J}'} \mu(B(J)) \| h_J \|_{C^\tau(B(J))} (1 + d_{B(J)}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_1)))^{-1/(2a^2+a^3)} .$$

Using Lemma 7.5.4, Lemma 7.5.11 and $a \geq 4$, we have that the above is bounded from above by

$$\leq 2^{540a^3} 2^{-Zn/(4a^2+2a^3)} \sum_{J \in \mathcal{J}'} \mu(B(J))$$

$$\times \prod_{j=1}^{2} (\inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_j)} g_j| + \inf_J M_{\mathcal{B},1} g_j) . \quad (7.5.22)$$

By the doubling property (1.0.5)

$$\mu(B(J)) \leq 2^{6a} \mu(B^\circ(J)) ,$$

thus

$$\mu(B(J)) \prod_{j=1}^{2} (\inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_j)} g_j| + \inf_J M_{\mathcal{B},1} g_j)$$

$$\leq 2^{6a} \int_{B^\circ(J)} \prod_{j=1}^{2} (|T^*_{\mathfrak{T}(\mathfrak{u}_j)} g_j|(x) + M_{\mathcal{B},1} g_j(x)) \, \mathrm{d}\mu(x)$$

$$\leq 2^{6a} \int_J \prod_{j=1}^{2} (|T^*_{\mathfrak{T}(\mathfrak{u}_j)} g_j|(x) + M_{\mathcal{B},1} g_j(x)) \, \mathrm{d}\mu(x) .$$

Summing over $J \in \mathcal{J}'$, we obtain

$$(7.5.22) \leq 2^{541a^3} 2^{-Zn/(4a^2+2a^3)} \int_X \prod_{j=1}^{2} (|T^*_{\mathfrak{T}(\mathfrak{u}_j)} g_j|(x) + M_{\mathcal{B},1} g_j(x)) \, \mathrm{d}\mu(x) .$$

Applying the Cauchy-Schwarz inequality, Lemma 7.4.5 follows. $\qquad \square$

7.6. Proof of The Remaining Tiles Lemma

We define

$$\mathcal{J}' = \{ J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}_1)) : J \subset \mathcal{J}(\mathfrak{u}_1) \} ,$$

note that this is different from the $\mathcal{J}'$ defined in the previous subsection.

LEMMA 7.6.1 (dyadic partition 2). *We have*

$$\mathcal{J}(\mathfrak{u}_1) = \dot{\bigcup_{J \in \mathcal{J}'}} J .$$

PROOF. By Lemma 7.1.2, it remains only to show that each $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))$ with $J \cap \mathcal{I}(\mathfrak{u}_1) \neq \varnothing$ is in $\mathcal{J}'$. But if $J \notin \mathcal{J}'$, then by (2.0.8) $\mathcal{I}(\mathfrak{u}_1) \subsetneq J$. Pick $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1)$. Then $\mathcal{I}(\mathfrak{p}) \subsetneq J$. This contradicts the definition of $\mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))$. $\qquad\square$

Lemma 7.4.6 follows from the following key estimate.

LEMMA 7.6.2 (bound for tree projection). *We have*

$$\|P_{\mathcal{J}'}|T^*_{\mathfrak{T}_2 \smallsetminus \mathfrak{S}} g_2|\|_2 \leq 2^{118a^3} 2^{-\frac{100}{202a} Zn\kappa} \|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} M_{\mathcal{B},1}|g_2|\|_2$$

We prove this lemma below. First, we deduce Lemma 7.4.6.

PROOF OF LEMMA 7.4.6. By Lemma 7.2.1 and Lemma 7.4.1, we have

$$(7.4.8) \leq 2^{104a^3} \|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}_1))}|g_1 \mathbf{1}_{\mathcal{J}(\mathfrak{u}_1)}|\|_2 \|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))}|T^*_{\mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}} g_2|\|_2 .$$

It follows from the definition of the projection operator P and Jensen's inequality that

$$\|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}_1))}|g_1 \mathbf{1}_{\mathcal{J}(\mathfrak{u}_1)}|\|_2 \leq \|g_1 \mathbf{1}_{\mathcal{J}(\mathfrak{u}_1)}\|_2 .$$

Since cubes in $\mathcal{J}'$ are pairwise disjoint and by Lemma 7.6.1, a cube $J \in \mathcal{J}'$ intersect $\mathcal{I}(\mathfrak{u}_1)$ if and only if $J \in \mathcal{J}'$. Thus

$$\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))}|T^*_{\mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}} g_2| = P_{\mathcal{J}'}|T^*_{\mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}} g_2| .$$

Combining this with Lemma 7.6.2, the definition (2.0.2) and $a \geq 4$ proves the lemma. $\qquad\square$

We need two more auxiliary lemmas before we prove Lemma 7.6.2.

LEMMA 7.6.3 (thin scale impact). *If* $\mathfrak{p} \in \mathfrak{T}_2 \smallsetminus \mathfrak{S}$ *and* $J \in \mathcal{J}'$ *with* $B(\mathcal{I}(\mathfrak{p})) \cap B(J) \neq \varnothing$, *then*

$$\mathrm{s}(\mathfrak{p}) \leq s(J) + 2 - \frac{Zn}{202a^3} .$$

PROOF. Suppose that $\mathrm{s}(\mathfrak{p}) > s(J) + 2 - \frac{Zn}{202a^3} =: s(J) - s_1$. Then, we have

$$\rho(\mathrm{c}(\mathfrak{p}), c(J)) \leq 8D^{s(J)} + 8D^{\mathrm{s}(\mathfrak{p})} \leq 16D^{\mathrm{s}(\mathfrak{p})+s_1} .$$

There exists a tile $\mathfrak{q} \in \mathfrak{T}(\mathfrak{u}_1)$. By (2.0.32), it satisfies $\mathcal{I}(\mathfrak{q}) \subsetneq \mathcal{I}(\mathfrak{u}_1)$. Thus $\mathcal{I}(\mathfrak{u}_1) \notin \mathcal{J}'$. It follows that $J \subsetneq \mathcal{I}(\mathfrak{u}_1)$. By (2.0.7) and (2.0.8), there exists a cube $J' \in \mathcal{D}$ with $J \subset J'$ and $s(J') = s(J) + 1$. By definition of $\mathcal{J}'$, there exists a tile $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}_1)$ with

$$\mathcal{I}(\mathfrak{p}') \subset B(c(J'), 100D^{s(J')+1}) .$$

By the triangle inequality, the definition (2.0.1) and $a \geq 4$, we have

$$B(c(J'), 100D^{s(J')+1}) \subset B(\mathrm{c}(\mathfrak{p}), 128D^{\mathrm{s}(\mathfrak{p})+s_1+1}) .$$

Since $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}_1)$ and $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$, we have by (2.0.36)

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{u}_2)) > 2^{Z(n+1)} .$$

Hence, by (2.0.32), the triangle inequality and using that by (2.0.3) $Z \geq 2$

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) > 2^{Z(n+1)} - 4 \geq 2^{Zn} .$$

It follows that

$$2^{Zn} \leq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \leq d_{B(\mathrm{c}(\mathfrak{p}), 128D^{\mathrm{s}(\mathfrak{p})+s_1+1})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) .$$

Using (1.0.8), we obtain

$$\leq 2^{9a+100a^3(s_1+1)} d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \, .$$

Since $\mathfrak{p}' \notin \mathfrak{S}$ this is bounded by

$$\leq 2^{9a+100a^3(s_1+1)} 2^{Zn/2} \, .$$

Thus

$$Zn/2 \leq 9a + 100a^3(s_1+1) \, ,$$

contradicting the definition of s_1. $\qquad\square$

LEMMA 7.6.4 (square function count). *For each $J \in \mathcal{J}'$, we have*

$$\frac{1}{\mu(J)} \int_J \left(\sum_{\substack{I \in \mathcal{D}, s(I)=s(J)-s \\ I \cap \mathcal{J}(\mathfrak{u}_1)=\varnothing \\ J \cap B(I) \neq \varnothing}} \mathbf{1}_{B(I)} \right)^2 \, \mathrm{d}\mu \leq 2^{104a^2} (8D^{-s})^{\kappa} \, .$$

PROOF OF LEMMA 7.6.4. Since $J \in \mathcal{J}'$ we have $J \subset \mathcal{J}(\mathfrak{u}_1)$. Thus, if $B(I) \cap J \neq \varnothing$ then

$$B(I) \cap J \subset \{ x \in J \ : \ \rho(x, X \smallsetminus J) \leq 8D^{s(I)} \} \, . \tag{7.6.1}$$

Furthermore, for each s the balls $B(I)$ with $s(I) = s$ have bounded overlap: Consider the collection $\mathcal{D}_{s,x}$ of all $I \in \mathcal{D}$ with $x \in B(I)$ and $s(I) = s$. By (2.0.10) and (2.0.8), the balls $B(c(I), \frac{1}{4}D^{s(I)})$, $I \in \mathcal{D}_{s,x}$ are disjoint, and by the triangle inequality, they are contained in $B(x, 9D^s)$. By the doubling property (1.0.5), we have

$$\mu(B(x, 9D^s)) \leq 2^{8a} \mu(B(c(I), \frac{1}{4}D^{s(I)}))$$

for each $I \in \mathcal{D}_{s,x}$. Thus

$$\mu(B(x, 9D^s)) \geq \sum_{I \in \mathcal{D}_{s,x}} \mu(B(c(I), \frac{1}{4}D^{s(I)})) \geq 2^{-8a} |\mathcal{D}_{s,x}| \mu(B(x, 9D^s)) \, .$$

Dividing by the positive $\mu(B(x, 9D^s))$, we obtain that for each x

$$\left(\sum_{\substack{I \in \mathcal{D}, s(I)=s(J)-s \\ I \cap \mathcal{J}(\mathfrak{u}_1)=\varnothing \\ J \cap B(I) \neq \varnothing}} \mathbf{1}_{B(I)}(x) \right)^2 = |\mathcal{D}_{s(J)-s,x}|^2 \leq 2^{16a} \, . \tag{7.6.2}$$

Combining (7.6.1), (7.6.2) and the small boundary property (2.0.11), noting that $8D^{s(I)} = 8D^{-s}D^{s(J)}$, the lemma follows. $\qquad\square$

PROOF OF LEMMA 7.6.2. Expanding the definition of $P_{\mathcal{J}'}$, we have

$$\|P_{\mathcal{J}'} | T^*_{\mathfrak{T}_2 \smallsetminus \mathfrak{S}} g_2 \|_2$$

$$= \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left| \int_J \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}} T^*_{\mathfrak{p}} g_2 \, \mathrm{d}\mu(y) \right|^2 \right)^{1/2} \, .$$

We split the innermost sum according to the scale of the tile $\mathfrak{p}$, and then apply the triangle inequality and Minkowski's inequality:

$$\leq \sum_{s=-S}^{S} \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left| \int_J \sum_{\substack{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S} \\ s(\mathfrak{p})=s}} T_{\mathfrak{p}}^* g_2 \, \mathrm{d}\mu(y) \right|^2 \right)^{1/2} .$$

By Lemma 7.4.1, the integral in the last display is 0 if $J \cap B(\mathcal{I}(\mathfrak{p})) = \varnothing$. By Lemma 7.6.3, it follows with $s_1 := \frac{Zn}{202a^3} - 2$:

$$= \sum_{s=s_1}^{s_1+2S} \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left| \int_J \sum_{\substack{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S} \\ s(\mathfrak{p})=s(J)-s \\ J \cap B(\mathcal{I}(\mathfrak{p})) \neq \varnothing}} T_{\mathfrak{p}}^* g_2 \, \mathrm{d}\mu(y) \right|^2 \right)^{1/2} . \qquad (7.6.3)$$

We have by Lemma 7.4.1 and (2.1.3)

$$\int |T_{\mathfrak{p}}^* g_2|(y) \, \mathrm{d}\mu(y)$$

$$\leq 2^{102a^3} \int_{B(\mathcal{I}(\mathfrak{p}))} \int \frac{1}{\mu(B(x, D^{s(\mathfrak{p})}))} \mathbf{1}_{E(\mathfrak{p})}(x) |g_2|(x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) .$$

If $x \in E(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p})$, then we have by (2.0.10) that

$$B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})}) \subset B(\mathcal{I}(\mathfrak{p})) .$$

Using the doubling property (1.0.5), it follows that

$$\mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})})) \leq 2^{3a} \mu(B(x, D^s)) .$$

Thus, using also $a \geq 4$

$$\int |T_{\mathfrak{p}}^* g_2|(y) \, \mathrm{d}\mu(y)$$

$$\leq 2^{103a^3} \int_{B(\mathcal{I}(\mathfrak{p}))} \int \frac{1}{\mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})}))} \mathbf{1}_{E(\mathfrak{p})}(x) |g_2|(x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) .$$

Since for each $I \in \mathcal{D}$ the sets $E(\mathfrak{p}), \mathfrak{p} \in \mathfrak{P}(I)$ are disjoint, it follows that

$$\left| \int_J \sum_{\substack{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S} \\ \mathcal{I}(\mathfrak{p})=I \\ J \cap B(\mathcal{I}(\mathfrak{p})) \neq \varnothing}} T_{\mathfrak{p}}^* g_2 \, \mathrm{d}\mu \right|$$

$$\leq 2^{103a^3} \int_J \mathbf{1}_{B(I)} \frac{1}{\mu(B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})}))} \int_{B(c(\mathfrak{p}), 4D^{s(\mathfrak{p})})} |g_2|(x) \, \mathrm{d}\mu(x)$$

$$\leq 2^{103a^3} \int_J M_{\mathcal{B}, 1} |g_2|(y) \mathbf{1}_{B(I)}(y) \, \mathrm{d}\mu(y) .$$

By Lemma 7.4.7, we have $\mathcal{J}(\mathfrak{p}) \cap \mathcal{J}(\mathfrak{u}_1) = \varnothing$ for all $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \smallsetminus \mathfrak{S}$. Thus we can estimate (7.6.3) by

$$2^{103a^3} \sum_{s=s_1}^{s_1+2S} \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left| \int_J \sum_{\substack{I \in \mathcal{D}, s(I)=s(J)-s \\ I \cap \mathcal{J}(\mathfrak{u}_1)=\varnothing \\ J \cap B(I) \neq \varnothing}} M_{\mathcal{B},1}|g_2| \mathbf{1}_{B(I)} \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}},$$

which is by Cauchy-Schwarz at most

$$2^{103a^3} \sum_{s=s_1}^{s_1+2S} \left(\sum_{J \in \mathcal{J}'} \int_J (M_{\mathcal{B},1}|g_2|)^2 \frac{1}{\mu(J)} \int_J \left(\sum_{\substack{I \in \mathcal{D}, s(I)=s(J)-s \\ I \cap \mathcal{J}(\mathfrak{u}_1)=\varnothing \\ J \cap B(I) \neq \varnothing}} \mathbf{1}_{B(I)} \right)^2 \mathrm{d}\mu \right)^{\frac{1}{2}}. \quad (7.6.4)$$

Using Lemma 7.6.4, we bound (7.6.4) by

$$2^{103a^3} \sum_{s=s_1}^{s_1+2S} \left(\sum_{J \in \mathcal{J}'} \int_J (M_{\mathcal{B},1}|g_2|)^2 2^{104a^2} (8D^{-s})^\kappa \right)^{\frac{1}{2}},$$

and, since dyadic cubes in $\mathcal{J}'$ form a partition of $\mathcal{J}(\mathfrak{u}_1)$ by Lemma 7.6.1, $\kappa \leq 1$ by (2.0.2), and $a \geq 4$

$$\leq 2^{116a^3} \sum_{s=s_1}^{s_1+2S} D^{-s\kappa/2} \|\mathbf{1}_{\mathcal{J}(\mathfrak{u}_1)} M_{\mathcal{B},1}|g_2|\|_2$$

$$\leq 2^{116a^3} D^{-s_1\kappa/2} \frac{1}{1-D^{-\kappa/2}} \|\mathbf{1}_{\mathcal{J}(\mathfrak{u}_1)} M_{\mathcal{B},1}|g_2|\|_2.$$

By convexity of $t \mapsto D^t$ and since $D \geq 2$, we have for all $-1 \leq t \leq 0$

$$D^t \leq 1 + t(1 - \frac{1}{D}) \leq 1 + \frac{1}{2}t.$$

Using this for $t = -\kappa/2$ and using that $s_1 = \frac{Zn}{202a^3} - 2$ and the definitions (2.0.1) and (2.0.2) of κ and D

$$\leq 2^{116a^3} 2^{-100a^2 (\frac{Zn}{202a^3} - 2)\frac{\kappa}{2}} \frac{2}{\kappa} \|\mathbf{1}_{\mathcal{J}(\mathfrak{u}_1)} M_{\mathcal{B},1}|g_2|\|_2$$

$$\leq \frac{2^{117a^3+1}}{\kappa} 2^{-\frac{100}{202a}Zn\kappa} \|\mathbf{1}_{\mathcal{J}(\mathfrak{u}_1)} M_{\mathcal{B},1}|g_2|\|_2.$$

Using the definition (2.0.2) of κ and $a \geq 4$, the lemma follows. $\square$

7.7. Forests

In this subsection, we complete the proof of Proposition 2.0.4 from the results of the previous subsections.

Define an n-row to be an n-forest $(\mathfrak{U}, \mathfrak{T})$, i.e. satisfying conditions (2.0.32) - (2.0.37), such that in addition the sets $\mathcal{J}(\mathfrak{u}), \mathfrak{u} \in \mathfrak{U}$ are pairwise disjoint.

LEMMA 7.7.1 (forest row decomposition). *Let $(\mathfrak{U}, \mathfrak{T})$ be an n-forest. Then there exists a decomposition*

$$\mathfrak{U} = \dot{\bigcup_{1 \leq j \leq 2^n}} \mathfrak{U}_j$$

such that for all $j = 1, \ldots, 2^n$ the pair $(\mathfrak{U}_j, \mathfrak{T}|_{\mathfrak{U}_j})$ is an n-row.

PROOF. Define recursively $\mathfrak{U}_j$ to be a maximal disjoint set of tiles $\mathfrak{u}$ in

$$\mathfrak{U} \smallsetminus \bigcup_{j' < j} \mathfrak{U}_{j'}$$

with inclusion maximal $\mathcal{I}(\mathfrak{u})$. Properties (2.0.32), -(2.0.37) for $(\mathfrak{U}_j, \mathfrak{T}|_{\mathfrak{U}_k})$ follow immediately from the corresponding properties for $(\mathfrak{U}, \mathfrak{T})$, and the cubes $\mathcal{I}(\mathfrak{u}), \mathfrak{u} \in \mathfrak{U}_j$ are disjoint by definition. The collections $\mathfrak{U}_j$ are also disjoint by definition.

Now we show by induction on j that each point is contained in at most $2^n - j$ cubes $\mathcal{I}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U} \smallsetminus \bigcup_{j' \leq j} \mathfrak{U}_{j'}$. This implies that $\bigcup_{j=1}^{2^n} \mathfrak{U}_j = \mathfrak{U}$, which completes the proof of the Lemma. For $j = 0$ each point is contained in at most 2^n cubes by (2.0.34). For larger j, if x is contained in any cube $\mathcal{I}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U} \smallsetminus \bigcup_{j' < j} \mathfrak{U}_{j'}$, then it is contained in a maximal such cube. Thus it is contained in a cube in $\mathcal{I}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U}_j$. Thus the number $\mathfrak{u} \in \mathfrak{U} \smallsetminus \bigcup_{j' \leq j} \mathfrak{U}_{j'}$ with $x \in \mathcal{I}(\mathfrak{u})$ is zero, or is less than the number of $\mathfrak{u} \in \mathfrak{U} \smallsetminus \bigcup_{j' \leq j-1} \mathfrak{U}_{j'}$ with $x \in \mathcal{I}(\mathfrak{u})$ by at least one. $\square$

We pick a decomposition of the forest $(\mathfrak{U}, \mathfrak{T})$ into 2^n n-rows

$$(\mathfrak{U}_j, \mathfrak{T}_j) := (\mathfrak{U}_j, \mathfrak{T}|_{\mathfrak{U}_j})$$

as in Lemma 7.7.1.

LEMMA 7.7.2 (row bound). *For each $1 \leq j \leq 2^n$ and each bounded g with bounded support, we have*

$$\left\| \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g \right\|_2 \leq 2^{156a^3} 2^{-n/2} \|g\|_2 \tag{7.7.1}$$

and

$$\left\| \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \mathbf{1}_F T_{\mathfrak{p}}^* g \right\|_2 \leq 2^{257a^3} 2^{-n/2} \operatorname{dens}_2 \left(\bigcup_{\mathfrak{u} \in \mathfrak{U}} \mathfrak{T}(\mathfrak{u}) \right)^{1/2} \|g\|_2 . \tag{7.7.2}$$

PROOF. By Lemma 7.3.1 and the density assumption (2.0.35), we have for each $\mathfrak{u} \in \mathfrak{U}$ and all bounded f of bounded support that

$$\left\| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \right\|_2 \leq 2^{155a^3} 2^{(4a+1-n)/2} \|f\|_2 \tag{7.7.3}$$

and

$$\left\| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} \mathbf{1}_F f \right\|_2 \leq 2^{256a^3} 2^{(4a+1-n)/2} \operatorname{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \|f\|_2 . \tag{7.7.4}$$

Since for each j the top cubes $\mathcal{J}(\mathfrak{u})$, $\mathfrak{u} \in \mathfrak{U}_j$ are disjoint, we further have for all bounded g of bounded support by Lemma 7.4.1

$$\left\| \mathbf{1}_F \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g \right\|_2^2 = \left\| \mathbf{1}_F \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \mathbf{1}_{\mathcal{J}(\mathfrak{u})} T_{\mathfrak{p}}^* \mathbf{1}_{\mathcal{J}(\mathfrak{u})} g \right\|_2^2$$

$$= \sum_{\mathfrak{u} \in \mathfrak{U}_j} \int_{\mathcal{J}(\mathfrak{u})} \left| \mathbf{1}_F \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* \mathbf{1}_{\mathcal{J}(\mathfrak{u})} g \right|^2 \mathrm{d}\mu \le \sum_{\mathfrak{u} \in \mathfrak{U}_j} \left\| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \mathbf{1}_F T_{\mathfrak{p}}^* \mathbf{1}_{\mathcal{J}(\mathfrak{u})} g \right\|_2^2 .$$

Applying the estimate for the adjoint operator following from equation (7.7.4), we obtain

$$\le 2^{256a^3} 2^{(4a+1-n)/2} \max_{\mathfrak{u} \in \mathfrak{U}_j} \mathrm{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \sum_{\mathfrak{u} \in \mathfrak{U}_j} \left\| \mathbf{1}_{\mathcal{J}(\mathfrak{u})} g \right\|_2^2 .$$

Again by disjointedness of the cubes $\mathcal{J}(\mathfrak{u})$, this is estimated by

$$2^{256a^3} 2^{(4a+1-n)/2} \max_{\mathfrak{u} \in \mathfrak{U}_j} \mathrm{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2^2 .$$

Thus, (7.7.2) follows, since $a \ge 4$. The proof of (7.7.1) from (7.7.3) is the same up to replacing F by X. $\qquad \square$

LEMMA 7.7.3 (row correlation). *For all $1 \le j < j' \le 2^n$ and for all bounded g_1, g_2 with bounded support, it holds that*

$$\left| \int \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{u}' \in \mathfrak{U}_{j'}} \sum_{\mathfrak{p} \in \mathfrak{T}_j(\mathfrak{u})} \sum_{\mathfrak{p}' \in \mathfrak{T}_{j'}(\mathfrak{u}')} T_{\mathfrak{p}}^* g_1 \overline{T_{\mathfrak{p}'}^* g_2} \, \mathrm{d}\mu \right| \le 2^{862a^3 - 3n} \|g_1\|_2 \|g_2\|_2 .$$

PROOF. To save some space we will write for subsets $\mathfrak{C} \subset \mathfrak{P}$

$$T_{\mathfrak{C}}^* = \sum_{\mathfrak{p} \in \mathfrak{C}} T_{\mathfrak{p}}^* .$$

We have by Lemma 7.4.1 and the triangle inequality that

$$\left| \int \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{u}' \in \mathfrak{U}_{j'}} \sum_{\mathfrak{p} \in \mathfrak{T}_j(\mathfrak{u})} \sum_{\mathfrak{p}' \in \mathfrak{T}_{j'}(\mathfrak{u}')} T_{\mathfrak{p}}^* g_1 \overline{T_{\mathfrak{p}'}^* g_2} \, \mathrm{d}\mu \right|$$

$$\le \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{u}' \in \mathfrak{U}_{j'}} \left| \int T_{\mathfrak{T}_j(\mathfrak{u})}^* (\mathbf{1}_{\mathcal{J}(\mathfrak{u})} g_1) \overline{T_{\mathfrak{T}_{j'}(\mathfrak{u}')}^* (\mathbf{1}_{\mathcal{J}(\mathfrak{u}')} g_2)} \, \mathrm{d}\mu \right| .$$

By Lemma 7.4.4, this is bounded by

$$2^{550a^3 - 3n} \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{u}' \in \mathfrak{U}_{j'}} \|S_{2,\mathfrak{u}} g_1\|_{L^2(\mathcal{J}(\mathfrak{u}') \cap \mathcal{J}(\mathfrak{u}))} \|S_{2,\mathfrak{u}'} g_2\|_{L^2(\mathcal{J}(\mathfrak{u}') \cap \mathcal{J}(\mathfrak{u}))} . \qquad (7.7.5)$$

We apply the Cauchy-Schwarz inequality in the form

$$\sum_{i \in M} a_i b_i \le \left(\sum_{i \in M} a_i^2 \right)^{1/2} \left(\sum_{i \in M} b_i^2 \right)^{1/2}$$

to the outer two sums:

$$\leq 2^{550a^3-3n} \left(\sum_{\mathfrak{u}\in\mathfrak{U}_j} \sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}} \|S_{2,\mathfrak{u}}g_1\|^2_{L^2(\mathcal{J}(\mathfrak{u}')\cap\mathcal{J}(\mathfrak{u}))} \right)^{1/2}$$

$$\left(\sum_{\mathfrak{u}\in\mathfrak{U}_j} \sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}} \|S_{2,\mathfrak{u}'}g_2\|^2_{L^2(\mathcal{J}(\mathfrak{u}')\cap\mathcal{J}(\mathfrak{u}))} \right)^{1/2}.$$

By pairwise disjointedness of the sets $\mathcal{J}(\mathfrak{u})$ for $\mathfrak{u} \in \mathfrak{U}_j$ and of the sets $\mathcal{J}(\mathfrak{u}')$ for $\mathfrak{u}' \in \mathfrak{U}_{j'}$, we have

$$\sum_{\mathfrak{u}\in\mathfrak{U}_j} \sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}} \|S_{2,\mathfrak{u}}g_1\|^2_{L^2(\mathcal{J}(\mathfrak{u}')\cap\mathcal{J}(\mathfrak{u}))} = \sum_{\mathfrak{u}\in\mathfrak{U}_j} \sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}} \int_{\mathcal{J}(\mathfrak{u})\cap\mathcal{J}(\mathfrak{u}')} |S_{2,\mathfrak{u}}g_1(y)|^2 \, \mathrm{d}\mu(y)$$

$$\leq \int_X |S_{2,\mathfrak{u}}g_1(y)|^2 \, \mathrm{d}\mu(y) = \|S_{2,\mathfrak{u}}g_1\|^2_2.$$

Arguing similar for g_2, we can estimate (7.7.5) to be

$$\leq 2^{550a^3-3n} \|S_{2,\mathfrak{u}}g_1\|_2 \|S_{2,\mathfrak{u}'}g_2\|_2.$$

The lemma now follows from Lemma 7.4.3. □

Define for $1 \leq j \leq 2^n$

$$E_j := \bigcup_{\mathfrak{u}\in\mathfrak{U}_j} \bigcup_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} E(\mathfrak{p}).$$

LEMMA 7.7.4 (disjoint row support). *The sets E_j, $1 \leq j \leq 2^n$ are pairwise disjoint.*

PROOF. Suppose that $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ and $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}')$ with $\mathfrak{u} \neq \mathfrak{u}'$ and $x \in E(\mathfrak{p})\cap E(\mathfrak{p}')$. Suppose without loss of generality that $\mathrm{s}(\mathfrak{p}) \leq \mathrm{s}(\mathfrak{p}')$. Then $x \in \mathcal{J}(\mathfrak{p}) \cap \mathcal{J}(\mathfrak{p}') \subset \mathcal{J}(\mathfrak{u}')$. By (2.0.8) it follows that $\mathcal{J}(\mathfrak{p}) \subset \mathcal{J}(\mathfrak{u}')$. By (2.0.36), it follows that

$$d_\mathfrak{p}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) > 2^{Z(n+1)}.$$

By the triangle inequality. Lemma 2.1.2 and (2.0.32) it follows that

$$d_\mathfrak{p}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}')) \geq d_\mathfrak{p}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) - d_\mathfrak{p}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{u}'))$$
$$> 2^{Z(n+1)} - d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{u}'))$$
$$\geq 2^{Z(n+1)} - 4.$$

Since $Z \geq 3$ by (2.0.3), it follows that $\mathcal{Q}(\mathfrak{p}') \notin B_\mathfrak{p}(\mathcal{Q}(\mathfrak{p}), 1)$, so $\Omega(\mathfrak{p}') \not\subset \Omega(\mathfrak{p})$ by (2.0.15). Hence, by (2.0.14), $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{p}') = \varnothing$. But if $x \in E(\mathfrak{p}) \cap E(\mathfrak{p}')$ then $Q(x) \in \Omega(\mathfrak{p}) \cap \Omega(\mathfrak{p}')$. This is a contradiction, and the lemma follows. □

Now we prove Proposition 2.0.4.

PROOF OF PROPOSITION 2.0.4. To save some space, we will write

$$T_{\mathfrak{R}_j}^* = \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^*.$$

By (7.4.1), we have for each j

$$T_{\mathfrak{R}_j}^* g = \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g = \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* \mathbf{1}_{E_j} g = T_{\mathfrak{R}_j}^* \mathbf{1}_{E_j} g.$$

Hence, by Lemma 7.7.1,

$$\left\| \sum_{\mathfrak{u} \in \mathfrak{U}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g \right\|_2^2 = \left\| \sum_{j=1}^{2^n} T_{\mathfrak{R}_j}^* g \right\|_2^2 = \left\| \sum_{j=1}^{2^n} T_{\mathfrak{R}_j}^* \mathbf{1}_{E_j} g \right\|_2^2$$

$$= \int_X \left| \sum_{j=1}^{2^n} T_{\mathfrak{R}_j}^* \mathbf{1}_{E_j} g \right|^2 \, d\mu$$

$$= \sum_{j=1}^{2^n} \int_X |T_{\mathfrak{R}_j}^* \mathbf{1}_{E_j} g|^2 + \sum_{j=1}^{2^n} \sum_{\substack{j'=1 \\ j' \neq j}}^{2^n} \int_X \overline{T_{\mathfrak{R}_j}^* \mathbf{1}_{E_j} g} \, T_{\mathfrak{R}_{j'}}^* \mathbf{1}_{E_{j'}} g \, d\mu.$$

We use Lemma 7.7.2 to estimate each term in the first sum, and Lemma 7.7.3 to bound each term in the second sum:

$$\leq 2^{312a^3 - n} \sum_{j=1}^{2^n} \|\mathbf{1}_{E_j} g\|_2^2 + 2^{862a^3 - 3n} \sum_{j=1}^{2^n} \sum_{j'=1}^{2^n} \|\mathbf{1}_{E_j} g\|_2 \|\mathbf{1}_{E_{j'}} g\|_2.$$

By Cauchy-Schwarz in the second two sums, this is at most

$$2^{862a^3} (2^{-n} + 2^n 2^{-3n}) \sum_{j=1}^{n} \|\mathbf{1}_{E_j} g\|_2^2,$$

and by disjointedness of the sets E_j, this is at most

$$2^{863a^3 - n} \|g\|_2^2.$$

Taking adjoints and square roots, it follows that for all f

$$\left\| \sum_{\mathfrak{u} \in \mathfrak{U}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \right\|_2 \leq 2^{432a^3 - \frac{n}{2}} \|f\|_2. \tag{7.7.6}$$

On the other hand, we have by disjointedness of the sets E_j from Lemma 7.7.4

$$\left\| \sum_{\mathfrak{u} \in \mathfrak{U}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \right\|_2^2 = \left\| \sum_{j=1}^{2^n} \mathbf{1}_{E_j} T_{\mathfrak{R}_j} f \right\|_2^2 = \sum_{j=1}^{2^n} \|\mathbf{1}_{E_j} T_{\mathfrak{R}_j} f\|_2^2.$$

If $|f| \leq \mathbf{1}_F$ then we obtain from Lemma 7.7.2 and taking square roots that

$$\leq 2^{257a^3} \operatorname{dens}_2\Big(\bigcup_{\mathfrak{u} \in \mathfrak{U}} \mathfrak{T}(\mathfrak{u}) \Big)^{\frac{1}{2}} 2^{-\frac{n}{2}} \Big(\sum_{j=1}^{2^n} \|f\|_2^2 \Big)^{\frac{1}{2}}$$

$$= 2^{257a^3} \operatorname{dens}_2(\bigcup_{\mathfrak{u} \in \mathfrak{U}} \mathfrak{T}(\mathfrak{u}))^{\frac{1}{2}} \|f\|_2 \,. \tag{7.7.7}$$

Proposition 2.0.4 follows by taking the product of the $(2 - \frac{2}{q})$-th power of (7.7.6) and the $(\frac{2}{q} - 1)$-st power of (7.7.7). $\qquad\square$

Proof of the Hölder cancellative condition

We need the following auxiliary lemma. Recall that $\tau = 1/a$.

LEMMA 8.0.1 (Lipschitz Holder approximation). *Let $z \in X$ and $R > 0$. Let $\varphi : X \to \mathbb{C}$ be a function supported in the ball $B := B(z, R)$ with finite norm $\|\varphi\|_{C^\tau(B)}$. Let $0 < t \le 1$. There exists a function $\tilde{\varphi} : X \to \mathbb{C}$, supported in $B(z, 2R)$, such that for every $x \in X$*

$$|\varphi(x) - \tilde{\varphi}(x)| \le t^\tau \|\varphi\|_{C^\tau(B)} \tag{8.0.1}$$

and

$$\|\tilde{\varphi}\|_{\mathrm{Lip}(B(z,2R))} \le 2^{4a} t^{-1-a} \|\varphi\|_{C^\tau(B)} \,. \tag{8.0.2}$$

PROOF. Define for $x, y \in X$ the Lipschitz and thus measurable function

$$L(x, y) := \max\{0, 1 - \frac{\rho(x, y)}{tR}\} \,. \tag{8.0.3}$$

We have that $L(x, y) \ne 0$ implies

$$y \in B(x, tR) \,. \tag{8.0.4}$$

We have for $y \in B(x, 2^{-1}tR)$ that

$$|L(x, y)| \ge 2^{-1} \,. \tag{8.0.5}$$

Hence

$$\int L(x, y)\, \mathrm{d}\mu(y) \ge 2^{-1} \mu(B(x, 2^{-1}Rt)) \,. \tag{8.0.6}$$

Let n be the smallest integer so that

$$2^n t \ge 1 \,. \tag{8.0.7}$$

Iterating $n + 2$ times the doubling condition (1.0.5), we obtain

$$\int L(x, y)\, \mathrm{d}\mu(y) \ge 2^{-1-a(n+2)} \mu(B(x, 2R)) \,. \tag{8.0.8}$$

Now define

$$\tilde{\varphi}(x) := \left(\int L(x, y)\, \mathrm{d}\mu(y) \right)^{-1} \int L(x, y) \varphi(y)\, \mathrm{d}\mu(y) \,.$$

Using that φ is supported in $B(z, R)$ and (8.0.4), we have that $\tilde{\varphi}$ is supported in $B(z, 2R)$.

We prove (8.0.1). For any $x \in X$, using that L is nonnegative,

$$\left(\int L(x, y)\, \mathrm{d}\mu(y) \right) |\varphi(x) - \tilde{\varphi}(x)| \tag{8.0.9}$$

$$= \left| \int L(x,y)(\varphi(x) - \varphi(y)) \, d\mu(y) \right| . \tag{8.0.10}$$

Using (8.0.4), we estimate the last display by

$$\leq \int_{B(x,tR)} L(x,y)|\varphi(x) - \varphi(y)| \, d\mu(y) . \tag{8.0.11}$$

Using the definition of $\|\varphi\|_{C^\tau(B)}$, we estimate the last display further by

$$\leq \left(\int_{B(x,tR)} L(x,y)\rho(x,y)^\tau \, d\mu(y) \right) \|\varphi\|_{C^\tau(B)} R^{-\tau} . \tag{8.0.12}$$

Using the condition on the domain of integration to estimate $\rho(x,y)$ by tR and then expanding the domain by positivity of the integrand, we estimate this further by

$$\leq \left(\int L(x,y) \, d\mu(y) \right) \|\varphi\|_{C^\tau(B)} t^\tau . \tag{8.0.13}$$

Dividing the string of inequalities from (8.0.9) to (8.0.13) by the positive integral of L proves (8.0.1).

We turn to (8.0.2). For every $x \in X$, we have

$$\left| \int L(x,y) \, d\mu(y) \right| |\tilde{\varphi}(x)| = \left| \int L(x,y)\varphi(y) \, d\mu(y) \right| \tag{8.0.14}$$

$$\leq \left| \int L(x,y) \, d\mu(y) \right| \sup_{x' \in X} |\varphi(x')| . \tag{8.0.15}$$

As φ is supported on B, dividing by the integral of L, we obtain

$$|\tilde{\varphi}(x)| \leq \sup_{x' \in B} |\varphi(x')| \leq \|\varphi\|_{C^\tau(B)} . \tag{8.0.16}$$

If $\rho(x,x') \geq R$, then we have by the triangle inequality

$$R \frac{|\tilde{\varphi}(x') - \tilde{\varphi}(x)|}{\rho(x,x')} \leq 2 \sup_{x'' \in X} |\tilde{\varphi}(x'')| \leq 2\|\varphi\|_{C^\tau(B)} . \tag{8.0.17}$$

Now assume $\rho(x,x') < R$. For $y \in X$ we have by the triangle inequality and a two fold case distinction for the maximum in the definition of L,

$$|L(x,y) - L(x',y)| \leq \frac{\rho(x,x')}{tR}. \tag{8.0.18}$$

We compute with (8.0.18), first adding and subtracting a term in the integral,

$$\left(\int L(x,y) \, d\mu(y) \right) |\tilde{\varphi}(x') - \tilde{\varphi}(x)| = \tag{8.0.19}$$

$$\left| \int L(x,y)\tilde{\varphi}(x') - L(x,y)\tilde{\varphi}(x) + L(x',y)\tilde{\varphi}(x') - L(x',y)\tilde{\varphi}(x') \, d\mu(y) \right| . \tag{8.0.20}$$

Grouping the second and third and the first and fourth term, we obtain using the definition of $\tilde{\varphi}$ and Fubini,

$$\leq \left| \int (L(x',y) - L(x,y))\varphi(y) \, d\mu(y) \right| \tag{8.0.21}$$

$$+ \left| \int L(x,y) \, \mathrm{d}\mu(y) - \int L(x',y) \, \mathrm{d}\mu(y) \right| |\tilde{\varphi}(x')| \qquad (8.0.22)$$

$$\leq 2 \int |L(x,y) - L(x',y)| \, \mathrm{d}\mu(y) \|\varphi\|_{C^\tau(B)}, \qquad (8.0.23)$$

where in the last inequality we have used (8.0.16). Using further (8.0.18) and the support of L, we estimate the last display by

$$\leq 2 \frac{\rho(x,x')}{tR} \mu(B(x,tR) \cup B(x',tR)) \|\varphi\|_{C^\tau(B)}. \qquad (8.0.24)$$

Using $\rho(x,x') < R$ and the triangle inequality, we estimate the last display by

$$\leq 2 \frac{\rho(x,x')}{tR} \mu(B(x,2R)) \|\varphi\|_{C^\tau(B)}. \qquad (8.0.25)$$

Dividing by the integral over L and using (8.0.8) and (8.0.7), we obtain

$$\frac{R|\tilde{\varphi}(x') - \tilde{\varphi}(x)|}{\rho(x,x')} \leq 2^{2+a(n+2)} t^{-1} \|\varphi\|_{C^\tau(B)} \leq 2^{2+3a} t^{-1-a} \|\varphi\|_{C^\tau(B)}. \qquad (8.0.26)$$

Combining (8.0.17) and (8.0.26) using $a \geq 4$ and $t \leq 1$ and adding (8.0.16) proves (8.0.2) and completes the proof of Lemma 8.0.1. $\qquad \square$

We turn to the proof of Proposition 2.0.5.

PROOF OF PROPOSITION 2.0.5. Let $z \in X$ and $R > 0$ and set $B = B(z,R)$. Let φ be given as in Proposition 2.0.5. Set

$$t := (1 + d_B(\vartheta,\theta))^{-\frac{\tau}{2+a}} \qquad (8.0.27)$$

and define $\tilde{\varphi}$ as in Lemma 8.0.1. Let ϑ and θ be in Θ. Then

$$\left| \int e(\vartheta(x) - \theta(x)) \varphi(x) \, \mathrm{d}\mu(x) \right| \qquad (8.0.28)$$

$$\leq \left| \int e(\vartheta(x) - \theta(x)) \tilde{\varphi}(x) \, \mathrm{d}\mu(x) \right| \qquad (8.0.29)$$

$$+ \left| \int e(\vartheta(x) - \theta(x))(\varphi(x) - \tilde{\varphi}(x)) \, \mathrm{d}\mu(x) \right| \qquad (8.0.30)$$

Using the cancellative condition (1.0.12) of Θ on the ball $B(z,2R)$, the term (8.0.29) is bounded above by

$$2^a \mu(B(z,2R)) \|\tilde{\varphi}\|_{\mathrm{Lip}(B(z,2R))} (1 + d_{B(z,2R)}(\vartheta,\theta))^{-\tau}. \qquad (8.0.31)$$

Using the doubling condition (1.0.5), the inequality (8.0.2), and the estimate $d_B \leq d_{B(z,2R)}$ from the definition, we estimate (8.0.31) from above by

$$2^{6a} t^{-1-a} \mu(B) \|\varphi\|_{C^\tau(B)} (1 + d_B(\vartheta,\theta))^{-\tau}. \qquad (8.0.32)$$

The term (8.0.30) we estimate using (8.0.1) and that ϑ and θ are real and thus $e(\vartheta)$ and $e(\theta)$ bounded in absolute value by 1. We obtain for (8.0.30) with (1.0.5) the upper bound

$$\mu(B(z,2R)) t^\tau \|\varphi\|_{C^\tau(B)} \leq 2^a \mu(B) t^\tau \|\varphi\|_{C^\tau(B)}. \qquad (8.0.33)$$

Using the definition (8.0.27) of t and adding (8.0.32) and (8.0.33) estimates (8.0.28) from above by

$$2^{6a}\mu(B)\|\varphi\|_{C^\tau(B)}(1 + d_B(\vartheta, \theta))^{-\frac{\tau}{2+a}} \tag{8.0.34}$$

$$+2^a\mu(B)\|\varphi\|_{C^\tau(B)}(1 + d_B(\vartheta, \theta))^{-\frac{\tau^2}{2+a}}. \tag{8.0.35}$$

$$\leq 2^{1+6a}\mu(B)\|\varphi\|_{C^\tau(B)}(1 + d_B(\vartheta, \theta))^{-\frac{\tau^2}{2+a}}, \tag{8.0.36}$$

where we used $\tau \leq 1$. This completes the proof of Proposition 2.0.5. $\qquad\square$

CHAPTER 9

Proof of Vitali covering and Hardy–Littlewood

We begin with a classical representation of the Lebesgue norm.

LEMMA 9.0.1 (layer cake representation). *Let $1 \leq p < \infty$. Then for any measurable function $u : X \to [0, \infty)$ on the measure space X relative to the measure μ we have*

$$\|u\|_p^p = p \int_0^\infty \lambda^{p-1} \mu(\{x : u(x) \geq \lambda\}) \, d\lambda \, . \tag{9.0.1}$$

PROOF. The left-hand side of (9.0.1) is by definition

$$\int_X u(x)^p \, d\mu(x) \, . \tag{9.0.2}$$

Writing $u(x)$ as an elementary integral in λ and then using Fubini, we write for the last display

$$= \int_X \int_0^{u(x)} p\lambda^{p-1} d\lambda \, d\mu(x) \tag{9.0.3}$$

$$= p \int_0^\infty \lambda^{p-1} \mu(\{x : u(x) \geq \lambda\}) d\lambda \, . \tag{9.0.4}$$

This proves the lemma. □

The following lemma will be used to define M in the proof of Proposition 2.0.6.

LEMMA 9.0.2 (covering separable space). *For each $r > 0$, there exists a countable collection $C(r) \subset X$ of points such that*

$$X \subset \bigcup_{c \in C(r)} B(c, r) \, .$$

PROOF. It clearly suffices to construct finite collections $C(r, k)$ such that

$$B(o, r2^k) \subset \bigcup_{c \in C(r,k)} B(c, r) \, ,$$

since then the collection $C(r) = \bigcup_{k \in \mathbb{N}} C(r, k)$ has the desired property.

Suppose that $Y \subset B(o, r2^k)$ is a collection of points such that for all $y, y' \in Y$ with $y \neq y'$, we have $\rho(y, y') \geq r$. Then the balls $B(y, r/2)$ are pairwise disjoint and contained in $B(o, r2^{k+1})$. If $y \in B(o, r)$, then $B(o, r2^{k+1}) \subset B(y, r2^{k+2})$. Thus, by the doubling property (1.0.5),

$$\mu(B(y, \frac{r}{2})) \geq 2^{-(k+2)a} \mu(B(o, r2^{k+1})) \, .$$

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Thus, we have

$$\mu(B(o, r2^{k+1})) \geq \sum_{y \in Y} \mu(B(y, \frac{r}{2})) \geq |Y| 2^{-(k+2)a} \mu(B(o, r2^{k+1})) \,.$$

We conclude that $|Y| \leq 2^{(k+2)a}$. In particular, there exists a set Y of maximal cardinality. Define $C(r, k)$ to be such a set.

If $x \in B(o, r2^k)$ and $x \notin C(r, k)$, then there must exist $y \in C(r, k)$ with $\rho(x, y) < r$. Thus $C(r, k)$ has the desired property. $\qquad\square$

We turn to the proof of Proposition 2.0.6.

PROOF OF PROPOSITION 2.0.6. Let the collection $\mathcal{B}$ be given. We first show (2.0.43).

We recursively choose a finite sequence $B_i \in \mathcal{B}$ for $i \geq 0$ as follows. Assume $B_{i'}$ is already chosen for $0 \leq i' < i$. If there exists a ball $B_i \in \mathcal{B}$ so that B_i is disjoint from all $B_{i'}$ with $0 \leq i' < i$, then choose such a ball $B_i = B(x_i, r_i)$ with maximal r_i.

If there is no such ball, stop the selection and set $i'' := i$.

By disjointedness of the chosen balls and since $0 \leq u$, we have

$$\sum_{0 \leq i < i''} \int_{B_i} u(x) \, d\mu(x) \leq \int_X u(x) \, d\mu(x) \,. \tag{9.0.5}$$

By (2.0.42), we conclude

$$\lambda \sum_{0 \leq i < i''} \mu(B_i) \leq \int_X u(x) \, d\mu(x) \,. \tag{9.0.6}$$

Let $x \in \bigcup \mathcal{B}$. Choose a ball $B' = B(x', r') \in \mathcal{B}$ such that $x \in B'$. If B' is one of the selected balls, then

$$x \in \bigcup_{0 \leq i < i''} B_i \subset \bigcup_{0 \leq i < i''} B(x_i, 3r_i) \,. \tag{9.0.7}$$

If B' is not one of the selected balls, then as it is not selected at time i'', there is a selected ball B_i with $B' \cap B_i \neq \emptyset$. Choose such B_i with minimal index i. As B' is therefore disjoint from all balls $B_{i'}$ with $i' < i$ and as it was not selected in place of B_i, we have $r_i \geq r'$.

Using a point y in the intersection of B_i and B', we conclude by the triangle inequality

$$\rho(x_i, x') \leq \rho(x_i, y) + \rho(x', y) \leq r_i + r' \leq 2r_i \,. \tag{9.0.8}$$

By the triangle inequality again, we further conclude

$$\rho(x_i, x) \leq \rho(x_i, x') + \rho(x', x) \leq 2r_i + r' \leq 3r_i \,. \tag{9.0.9}$$

It follows that

$$x \in \bigcup_{0 \leq i < i''} B(x_i, 3r_i) \,. \tag{9.0.10}$$

With (9.0.7) and (9.0.10), we conclude

$$\bigcup \mathcal{B} \subset \bigcup_{0 \leq i < i''} B(x_i, 3r_i) \,. \tag{9.0.11}$$

With the doubling property (1.0.5) applied twice, we conclude

$$\mu(\bigcup \mathcal{B}) \leq \sum_{0 \leq i < i''} \mu(B(x_i, 3r_i)) \leq 2^{2a} \sum_{0 \leq i < i''} \mu(B_i). \tag{9.0.12}$$

With (9.0.6) and (9.0.12) we conclude (2.0.43).

We turn to the proof of (2.0.44). We first consider the case $p_1 = 1$ and recall $M_{\mathcal{B}} = M_{\mathcal{B},1}$. We write for the p_2-th power of left-hand side of (2.0.44) with Lemma 9.0.1 and a change of variables

$$\|M_{\mathcal{B}}u(x)\|_{p_2}^{p_2} = p_2 \int_0^\infty \lambda^{p_2-1} \mu(\{x : M_{\mathcal{B}}u(x) \geq \lambda\}) d\lambda \tag{9.0.13}$$

$$= 2^{p_2} p_2 \int_0^\infty \lambda^{p_2-1} \mu(\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\}) d\lambda. \tag{9.0.14}$$

Fix $\lambda \geq 0$ and let $x \in X$ satisfy $M_{\mathcal{B}}u(x) \geq 2\lambda$. By definition of $M_{\mathcal{B}}$, there is a ball $B' \in \mathcal{B}$ such that $x \in B'$ and

$$\int_{B'} u(y) \, d\mu(y) \geq 2\lambda\mu(B'). \tag{9.0.15}$$

Define $u_\lambda(y) := 0$ if $|u(y)| < \lambda$ and $u_\lambda(y) := u(y)$ if $|u(y)| \geq \lambda$. Then with (9.0.15)

$$\int_{B'} u_\lambda(y) \, d\mu(y) = \int_{B'} u(y) \, d\mu(y) - \int_{B'} (u - u_\lambda)(y) d\mu(y) \tag{9.0.16}$$

$$\geq 2\lambda\mu(B') - \int_{B'} (u - u_\lambda)(y) d\mu(y). \tag{9.0.17}$$

As $(u - u_\lambda)(y) \leq \lambda$ by definition, we can estimate the last display by

$$\geq 2\lambda\mu(B') - \int_{B'} \lambda \, d\mu(y) = \lambda\mu(B'). \tag{9.0.18}$$

Hence x is contained in $\bigcup(\mathcal{B}_\lambda)$, where $\mathcal{B}_\lambda$ is the collection of balls B'' in $\mathcal{B}$ such that

$$\int_{B''} u_\lambda(y) \, d\mu(y) \geq \lambda\mu(B''). \tag{9.0.19}$$

We have thus seen

$$\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\} \subset \bigcup \mathcal{B}_\lambda. \tag{9.0.20}$$

Applying (2.0.43) to the collection $\mathcal{B}_\lambda$ gives

$$\lambda\mu(\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\}) \leq 2^{2a} \int u_\lambda(x) \, dx. \tag{9.0.21}$$

With Lemma 9.0.1,

$$\lambda\mu(\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\}) \leq 2^{2a} \int_0^\infty \mu(\{x : |u_\lambda(x)| \geq \lambda'\}) \, d\lambda'. \tag{9.0.22}$$

By definition of u_λ, making a case distinction between $\lambda \geq \lambda'$ and $\lambda < \lambda'$, we see that

$$\mu(\{x : |u_\lambda(x)| \geq \lambda'\}) \leq \mu(\{x : |u(x)| \geq \max(\lambda, \lambda')\}). \tag{9.0.23}$$

We obtain with (9.0.14), (9.0.22), and (9.0.23)

$$\|M_{\mathcal{B}}u(x)\|_{p_2}^{p_2} \tag{9.0.24}$$

$$\leq 2^{p_2+2a} p_2 \int_0^\infty \lambda^{p_2-2} \int_0^\infty \mu(\{x : |u(x)| \geq \max(\lambda, \lambda')\}) \, d\lambda' d\lambda. \qquad (9.0.25)$$

We split the integral into $\lambda \geq \lambda'$ and $\lambda < \lambda'$ and resolve the maximum correspondingly. We have for $\lambda \geq \lambda'$ with Lemma 9.0.1

$$\int_0^\infty \lambda^{p_2-2} \int_0^\lambda \mu(\{x : |u(x)| \geq \lambda\}) \, d\lambda' d\lambda \qquad (9.0.26)$$

$$= \int_0^\infty \lambda^{p_2-1} \mu(\{x : |u(x)| \geq \lambda\}) d\lambda. \qquad (9.0.27)$$

$$= p_2^{-1} \|u\|_{p_2}^{p_2}. \qquad (9.0.28)$$

We have for $\lambda < \lambda'$ with Fubini and Lemma 9.0.1

$$\int_0^\infty \lambda^{p_2-2} \int_\lambda^\infty \mu(\{x : |u(x)| \geq \lambda'\}) \, d\lambda' d\lambda. \qquad (9.0.29)$$

$$= \int_0^\infty \int_0^{\lambda'} \lambda^{p_2-2} \mu(\{x : |u(x)| \geq \lambda'\}) d\lambda d\lambda'. \qquad (9.0.30)$$

$$= (p_2-1)^{-1} \int_0^\infty (\lambda')^{p_2-1} \mu(\{x : |u(x)| \geq \lambda'\}) d\lambda'. \qquad (9.0.31)$$

$$= (p_2-1)^{-1} p_2^{-1} \|u\|_{p_2}^{p_2}. \qquad (9.0.32)$$

Adding the two estimates (9.0.28) and (9.0.32) gives

$$\|M_{\mathcal{B}} u(x)\|_{p_2}^{p_2} \leq 2^{p_2+2a} (1 + (p_2-1)^{-1}) \|u\|_{p_2}^{p_2} = 2^{p_2+2a} p_2 (p_2-1)^{-1} \|u\|_{p_2}^{p_2}. \qquad (9.0.33)$$

With $a \geq 1$ and $p_2 > 1$, taking the p_2-th root, we obtain (2.0.44). We turn to the case of general $1 \leq p_1 < p_2$. We have

$$M_{\mathcal{B}, p_1} u = (M_{\mathcal{B}}(|u|^{p_1}))^{\frac{1}{p_1}}. \qquad (9.0.34)$$

Applying the special case of (2.0.44) for $M_{\mathcal{B}}$ gives

$$\|M_{\mathcal{B}, p_1} u\|_{p_2} = \|M_{\mathcal{B}}(|u|^{p_1})\|_{p_2/p_1}^{\frac{1}{p_1}} \qquad (9.0.35)$$

$$\leq 2^{2a} (p_2/p_1)(p_2/p_1 - 1)^{-1} \||u|^{p_1}\|_{p_2/p_1}^{\frac{1}{p_1}} = 2^{2a} p_2 (p_2-p_1)^{-1} \|u\|_{p_2}. \qquad (9.0.36)$$

This proves (2.0.44) in general.

Now we construct the operator M satisfying (2.0.45) and (2.0.46). For each $k \in \mathbb{Z}$ we choose a countable set $C(2^k)$ as in Lemma 9.0.2. Define

$$\mathcal{B}_\infty = \{B(c, 2^k) \ : \ c \in C(2^k), k \in \mathbb{Z}\}.$$

By Lemma 9.0.2, this is a countable collection of balls. We choose an enumeration $\mathcal{B}_\infty = \{B_1, \dots\}$ and define

$$\mathcal{B}_n = \{B_1, \dots, B_n\}.$$

We define

$$Mw := 2^{2a} \sup_{n \in \mathbb{N}} M_{\mathcal{B}_n} w.$$

This function is measurable for each measurable w, since it is a countable supremum of measurable functions. Estimate (2.0.46) follows immediately from (2.0.44) and the monotone convergence theorem.

It remains to show (2.0.45). Let $B = B(x, r) \subset X$. Let k be the smallest integer such that $2^k \geq r$, in particular we have $2^k < 2r$. By definition of $C(2^k)$, there exists $c \in C(2^k)$ with $x \in B(c, 2^k)$. By the triangle inequality, we have $B(c, 2^k) \subset B(x, 4r)$, and hence by the doubling property (1.0.5)

$$\mu(B(c, 2^k)) \leq 2^{2a} \mu(B(x, r)).$$

It follows that for each $z \in B(x, r)$

$$\frac{1}{\mu(B(x, r))} \int_{B(x,r)} |w(y)| \, \mathrm{d}\mu(y) \leq \frac{2^{2a}}{\mu(B(c, 2^k))} \int_{B(c,2^k)} |w(y)| \, \mathrm{d}\mu(y)$$
$$\leq Mw(z).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Two-sided Metric Space Carleson

We prove a variant of Theorem 1.0.2 for a two-sided Calderón–Zygmund kernel on the doubling metric measure space (X, ρ, μ, a), i.e. a one-sided Calderón–Zygmund kernel K which additionally satisfies for all $x, x', y \in X$ with $x \neq y$ and $2\rho(x, x') \leq \rho(x, y)$,

$$|K(x, y) - K(x', y)| \leq \left(\frac{\rho(x, x')}{\rho(x, y)} \right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x, y)}. \tag{10.0.1}$$

By the additional regularity, we can weaken the assumption (1.0.18) to an operator that is easier to work with in applications. Namely, for $r > 0$, $x \in X$, and a bounded, measurable function $f : X \to \mathbb{C}$ supported on a set of finite measure, we define

$$T_r f(x) := \int_{r \leq \rho(x, y)} K(x, y) f(y) \, d\mu(y) = \int_{X \smallsetminus B(x, r)} K(x, y) f(y) \, d\mu(y). \tag{10.0.2}$$

THEOREM 10.0.1 (two-sided metric space Carleson). *For all integers $a \geq 4$ and real numbers $1 < q \leq 2$ the following holds. Let (X, ρ, μ, a) be a doubling metric measure space. Let Θ be a cancellative compatible collection of functions and let K be a two-sided Calderón–Zygmund kernel on (X, ρ, μ, a). Assume that for every bounded measurable function g on X supported on a set of finite measure and all $r > 0$ we have*

$$\|T_r g\|_2 \leq 2^{a^3} \|g\|_2. \tag{10.0.3}$$

Then for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have, with T defined in (1.0.17),

$$\left| \int_G T f \, d\mu \right| \leq \frac{2^{452a^3}}{(q-1)^6} \mu(G)^{1 - \frac{1}{q}} \mu(F)^{\frac{1}{q}}. \tag{10.0.4}$$

For the remainder of this chapter, fix an integer $a \geq 4$, a doubling metric measure space (X, ρ, μ, a) and a two-sided Calderón–Zygmund kernel K as in Theorem 10.0.1.

The following lemma is proved in Section 10.1.

LEMMA 10.0.2 (nontangential-from-simple). *Assume (10.0.3) holds. Then, for every bounded measurable function $g : X \to \mathbb{C}$ supported on a set of finite measure we have*

$$\|T_* g\|_2 \leq 2^{3a^3} \|g\|_2. \tag{10.0.5}$$

PROOF OF THEOREM 10.0.1. Let $1 < q \leq 2$ be a real number. Let Θ be a cancellative compatible collection of functions. By the assumption (10.0.3), we can

apply Lemma 10.0.2 to obtain

$$\|T_* g\|_2 \leq 2^{3a^3} \|g\|_2. \tag{10.0.6}$$

Define

$$K'(x, y) := 2^{-2a^3} K(x, y).$$

Then K' is a two-sided Calderón–Zygmund kernel on (X, ρ, μ, a). Denote the corresponding maximally truncated non-tangential singular operator by T'_* and the corresponding generalized Carleson operator by T'. With (10.0.6), we obtain

$$\|T'_* g\|_2 \leq 2^{a^3} \|g\|_2. \tag{10.0.7}$$

Applying Theorem 1.0.2 for K' yields that for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have

$$\left| \int_G T' f \, \mathrm{d}\mu \right| \leq \frac{2^{450a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

This finishes the proof since for all $x \in X$,

$$T' f(x) = 2^{-2^a} T f(x).$$

$\square$

The proof of Lemma 10.0.2 relies on the following auxiliary lemma which is proved in Section 10.2.

LEMMA 10.0.3 (Weak 1 1). *Let $f : X \to \mathbb{C}$ be a bounded measurable function supported on a set of finite measure and assume for some $r > 0$ that for every bounded measurable function g on X supported on a set of finite measure,*

$$\|T_r g\|_2 \leq 2^{a^3} \|g\|_2. \tag{10.0.8}$$

Then for all $\alpha > 0$, we have

$$\mu \left(\{ x \in X : |T_r f(x)| > \alpha \} \right) \leq \frac{2^{a^3 + 19a}}{\alpha} \int |f(y)| \, d\mu(y). \tag{10.0.9}$$

Throughout Sections 10.1 and 10.2, for any measurable bounded function $w : X \to \mathbb{C}$, let $Mw : X \to [0, \infty)$ denote the corresponding Hardy–Littlewood maximal function defined in Proposition 2.0.6. Apart from Proposition 2.0.6, Sections 10.1 and 10.2 have no dependencies in the previous chapters.

10.1. Proof of Cotlar's Inequality

LEMMA 10.1.1. *For all real numbers $a \geq 4$,*

$$\sum_{n=0}^{\infty} 2^{-\frac{n}{a}} \leq 2^a.$$

PROOF. By convexity, for all $0 \leq \lambda \leq 1$

$$2^{\lambda(-\frac{1}{4})} \leq \lambda 2^{-\frac{1}{4}} + (1 - \lambda) 2^0.$$

For $\lambda := \frac{4}{a}$, we obtain

$$2^{-\frac{1}{a}} \leq 1 - (1 - 2^{-\frac{1}{4}}) \frac{4}{a}.$$

We conclude

$$\sum_{n=0}^{\infty} 2^{-\frac{n}{a}} = \frac{1}{1 - 2^{-\frac{1}{a}}} \leq \frac{1}{4(1 - 2^{-\frac{1}{4}})} a \leq 2^a.$$

$\square$

LEMMA 10.1.2 (estimate x shift). *Let $0 < r$ and $x \in X$. Let $g : X \to \mathbb{C}$ be a bounded measurable function supported on a set of finite measure. Then for all x' with $\rho(x, x') \leq r$.*

$$|T_r g(x) - T_r g(x')| \leq 2^{a^3 + 2a + 2} M g(x).$$

PROOF. By definition,

$$|T_r g(x) - T_r g(x')| = \left| \int_{r \leq \rho(x,y)} K(x,y) g(y) \, d\mu(y) - \int_{r \leq \rho(x',y)} K(x',y) g(y) \, d\mu(y) \right|. \tag{10.1.1}$$

We split the first integral in (10.1.1) into the domains $r \leq \rho(x,y) < 2r$ and $2r \leq \rho(x,y)$. The integral over the first domain we estimate by (10.1.2) below. For the second domain, we observe with $\rho(x, x') \leq r$ and the triangle inequality that $r \leq \rho(x', y)$. We therefore combine on this domain with the corresponding part of the second integral in (10.1.1) and estimate that by (10.1.3) below. The remaining part of the second integral in (10.1.1) we estimate by (10.1.4). Overall, we have estimated (10.1.1) by

$$\int_{r \leq \rho(x,y) < 2r} |K(x,y)||g(y)| \, d\mu(y) \tag{10.1.2}$$

$$+ \left| \int_{2r \leq \rho(x,y)} (K(x',y) - K(x,y)) g(y) \, d\mu(y) \right| \tag{10.1.3}$$

$$+ \left| \int_{r \leq \rho(x',y), r \leq \rho(x,y) < 2r} |K(x',y) g(y)| \, d\mu(y) \right| \tag{10.1.4}$$

Using the bound on K in (1.0.14) and the doubling condition (1.0.5), we estimate (10.1.2) by

$$\int_{r \leq \rho(x,y) < 2r} \frac{2^{a^3}}{V(x,y)} |g(y)| \, d\mu(y) \leq \frac{2^{a^3}}{\mu(B(x,r))} \int_{r \leq \rho(x,y) < 2r} |g(y)| \, d\mu(y) \tag{10.1.5}$$

$$\leq \frac{2^{a^3} \cdot 2^a}{\mu(B(x,2r))} \int_{\rho(x,y) < 2r} |g(y)| \, d\mu(y). \tag{10.1.6}$$

Using the definition of Mg, we estimate (10.1.6) by

$$\leq 2^{a^3 + a} M g(x). \tag{10.1.7}$$

Similarly, in the domain of (10.1.4) we note by the triangle inequality and assumption on x' that $\rho(x', y) < 3r$ and thus we estimate (10.1.4) by

$$\frac{2^{a^3}}{\mu(B(x',r))} \int_{\rho(x',y) < 4r} |g(y)| \, d\mu(y) \leq 2^{a^3 + 2a} M g(x) \tag{10.1.8}$$

We turn to the remaining term. Using (10.0.1), we estimate (10.1.3) by

$$\int_{2r\leq\rho(x,y)} \left(\frac{\rho(x,x')}{\rho(x,y)}\right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x,y)}|g(y)|\,d\mu(y) \tag{10.1.9}$$

We decompose and estimate (10.1.3) with the triangle inequality by

$$\sum_{j=1}^{\infty}\int_{2^jr\leq\rho(x,y)<2^{j+1}r} \left(\frac{\rho(x,x')}{\rho(x,y)}\right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x,y)}|g(y)|\,d\mu(y) \tag{10.1.10}$$

$$\leq \sum_{j=1}^{\infty}\left(2^{-j}\right)^{\frac{1}{a}}\int_{2^jr\leq\rho(x,y)<2^{j+1}r} \frac{2^{a^3}}{\mu(B(x,2^jr))}|g(y)|\,d\mu(y) \tag{10.1.11}$$

$$\leq \sum_{j=1}^{\infty}2^{-\frac{j}{a}}\frac{2^{a^3+a}}{\mu(B(x,2^{j+1}r))}\int_{\rho(x,y)<2^{j+1}r}|g(y)|\,d\mu(y) \tag{10.1.12}$$

$$\leq 2^{a^3+a}\sum_{j=1}^{\infty}2^{-\frac{j}{a}}Mg(x)\,. \tag{10.1.13}$$

Using Lemma 10.1.1, we estimate (10.1.13) by

$$\leq 2^{a^3+2a}Mg(x)\,. \tag{10.1.14}$$

Summing the estimates for (10.1.2), (10.1.3), and (10.1.4) proves the lemma. $\qquad\square$

LEMMA 10.1.3 (Cotlar control). *Let $0 < r \leq R$ and $x \in X$. Let $g : X \to \mathbb{C}$ be a bounded measurable function supported on a set of finite measure. Then for all $x' \in X$ with $\rho(x,x') \leq \frac{R}{4}$ we have*

$$|T_Rg(x)| \leq |T_r(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x')| + 2^{a^3+4a+1}Mg(x)\,. \tag{10.1.15}$$

PROOF. Let x and x' be given with $\rho(x,x') \leq \frac{R}{4}$. By an application of Lemma 10.1.2, we estimate the left-hand-side of (10.1.15) by

$$|T_R(g)(x')| + 2^{a^3+2a+2}Mg(x)\,. \tag{10.1.16}$$

We have

$$T_R(g)(x') = \int_{R\leq\rho(x',y)}K(x',y)g(y)\,d\mu(y)\,. \tag{10.1.17}$$

On the domain $R \leq \rho(x',y)$, we have $\frac{R}{2} \leq \rho(x,y)$. Hence we may write for (10.1.17)

$$T_R(g)(x') = \int_{R\leq\rho(x',y)}K(x',y)(g - g\mathbf{1}_{B(x,\frac{R}{2})})(y)\,d\mu(y)$$

$$= T_R(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x')\,. \tag{10.1.18}$$

Combining the estimate (10.1.16) with the identification (10.1.18), we obtain

$$|T_Rg(x)| \leq |T_R(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x')| + 2^{a^3+2a+2}Mg(x)\,. \tag{10.1.19}$$

We have

$$(T_r - T_R)(g - g\mathbf{1}_{B(x,\frac{R}{2})})(x')$$

$$= \int_{B(x',R)\smallsetminus B(x',r)} K(x',y)(g - g\mathbf{1}_{B(x,\frac{R}{2})})(y)\, d\mu(y)$$

$$= \int_{B(x',R)\smallsetminus(B(x',r)\cup B(x,\frac{R}{2}))} K(x',y)g(y)\, d\mu(y) \qquad (10.1.20)$$

As $\frac{R}{2} \le \rho(x,y)$ together with $\rho(x,x') \le \frac{R}{4}$ implies $\frac{R}{4} \le \rho(x',y)$, we can estimate (10.1.20) with (1.0.14) by

$$\le \frac{2^{a^3}}{\mu(B(x',\frac{R}{4}))} \int_{B(x,2R)\smallsetminus B(x',\frac{R}{4})} |g(y)|\, d\mu(y)$$

$$\le \frac{2^{a^3+a}}{\mu(B(x',\frac{R}{2}))} \int_{B(x,2R)} |g(y)|\, d\mu(y)$$

$$\le 2^{a^3+a} \frac{\mu(B(x,2R))}{\mu(B(x,\frac{R}{4}))} Mg(x) \le 2^{a^3+4a} Mg(x)\,.$$

By the triangle inequality, (10.1.15) follows now from (10.1.19) and the estimate for (10.1.20). $\qquad\qquad\square$

LEMMA 10.1.4 (Cotlar sets). *Assume that* (10.0.3) *holds. Let* $0 < r \le R$ *and* $x \in X$. *Let* $g : X \to \mathbb{C}$ *be a bounded measurable function supported on a set of finite measure. Then the measure* $|F_1|$ *of the set* F_1 *of all* $x' \in B(x, \frac{R}{4})$ *such that*

$$|T_r g(x')| > 4M(T_r g)(x) \qquad (10.1.21)$$

is less than or equal to $\mu(B(x,\frac{R}{4}))/4$. *Moreover, the measure* $|F_2|$ *of the set* F_2 *of all* $x' \in B(x, \frac{R}{4})$ *such that*

$$|T_r(g\mathbf{1}_{B(x,\frac{R}{2})})(x')| > 2^{a^3+20a+2} Mg(x) \qquad (10.1.22)$$

is less than or equal to $\mu(B(x,\frac{R}{4}))/4$.

PROOF. Let r, R, x and g be given. If $M(T_r g)(x) = 0$, then $T_r g$ is zero almost everywhere and the estimate on $|F_1|$ is trivial. Assume $M(T_r g)(x) > 0$. We have with (10.1.21)

$$M(T_r g)(x) \ge \frac{1}{\mu(B(x,\frac{R}{4}))} \int_{B(x,\frac{R}{4})} |T_r g(x')|\, dx' \qquad (10.1.23)$$

$$\ge \frac{1}{\mu(B(x,\frac{R}{4}))} \int_{F_1} 4M(T_r g)(x)\, dx' \qquad (10.1.24)$$

Dividing by $M(T_r g)(x)$ gives

$$1 \ge \frac{4}{\mu(B(x,\frac{R}{4}))} |F_1|\,. \qquad (10.1.25)$$

This gives the desired bound for the measure of F_1. We turn to the set F_2. Similarly as above we may assume $Mg(x) > 0$. The set F_2 is then estimated with Lemma 10.0.3 by

$$\frac{2^{a^3+19a}}{2^{a^3+20a+2} Mg(x)} \int |g\mathbf{1}_{B(x,\frac{R}{2})}|(y)\, d\mu(y) \qquad (10.1.26)$$

$$\leq \frac{1}{2^{a+2}Mg(x)}\mu(B(x,\frac{R}{2}))Mg(x) \leq \frac{\mu(B(x,\frac{R}{4}))}{4}\,. \qquad (10.1.27)$$

This gives the desired bound for the measure of F_2. □

LEMMA 10.1.5 (Cotlar estimate). *Assume that* (10.0.3) *holds. Let* $0 < r \leq R$ *and* $x \in X$. *Let* $g : X \to \mathbb{C}$ *be a bounded measurable function supported on a set of finite measure. Then*

$$|T_R g(x)| \leq 2^2 M(T_r g)(x) + 2^{a^3+20a+3}Mg(x)\,. \qquad (10.1.28)$$

PROOF. By Lemma 10.1.4, the set of all $x' \in B(x,\frac{R}{4})$ such that at least one of the conditions (10.1.21) and (10.1.22) is satisfied has measure less than or equal to $\mu(B(x,\frac{R}{4}))/2$ and hence is not all of $B(x,\frac{R}{4})$. Pick an $x' \in B(x,\frac{R}{4})$ such that both conditions are not satisfied. Applying Lemma 10.1.3 for this x' and using the triangle inequality estimates the left-hand side of (10.1.28) by

$$4M(T_r g)(x) + 2^{a^3+20a+2}Mg(x) + 2^{a^3+4a+1}Mg(x)\,. \qquad (10.1.29)$$

This proves the lemma. □

LEMMA 10.1.6 (simple nontangential operator). *Assume that* (10.0.3) *holds. For every* $r > 0$ *and every bounded measurable function g supported on a set of finite measure we have*

$$\|T_*^r g\|_2 \leq 2^{a^3+24a+6}\|g\|_2, \qquad (10.1.30)$$

where

$$T_*^r g(x) := \sup_{r<R}\sup_{x'\in B(x,R)} |T_R(g)(x')|\,. \qquad (10.1.31)$$

PROOF. With Lemma 10.1.2 and the triangle inequality, we estimate for every $x \in X$

$$T_*^r g(x) \leq 2^{a^3+2a+2}Mg(x) + \sup_{r<R}|T_R(g)(x)|\,. \qquad (10.1.32)$$

Using further Lemma 10.1.5, we estimate

$$T_*^r g(x) \leq 2^{a^3+2a+2}Mg(x) + 2^{a^3+20a+3}Mg(x) + 2^2 M(T_r g)(x)\,. \qquad (10.1.33)$$

Taking the L^2 norm and using Proposition 2.0.6 with $a = 4$ and $p_2 = 2$ and $p_1 = 1$, we obtain

$$\|T_*^r g\|_2 \leq 2^{a^3+20a+4}\|Mg\|_2 + 2^2\|M(T_r g)\|_2 \qquad (10.1.34)$$

$$\leq 2^{a^3+24a+5}\|g\|_2 + 2^{4a+3}\|T_r(g)\|_2\,. \qquad (10.1.35)$$

Applying (10.0.3), gives

$$\|T_*^r g\|_2 \leq 2^{a^3+24a+5}\|g\|_2 + 2^{a^3+4a+3}\|g\|_2\,. \qquad (10.1.36)$$

This shows (10.1.30) and completes the proof of the lemma. □

In order to pass from the one-sided truncation in T_r and T_*^r to the two-sided truncation in T_*, we show in the following two lemmas that the integral in (1.0.16) can be exchanged for an integral over the difference of two balls.

LEMMA 10.1.7. *Let f be a bounded measurable function with bounded support on* X. *Let $x \in X$ and $R > 0$. Then, for all $\epsilon > 0$, there exists some $\delta > 0$ such that*

$$\left| \int_{R < \rho(x,y) < R+\delta} K(x,y) f(y) \, d\mu(y) \right| \leq \epsilon \tag{10.1.37}$$

and

$$\left| \int_{R-\delta < \rho(x,y) < R} K(x,y) f(y) \, d\mu(y) \right| \leq \epsilon. \tag{10.1.38}$$

PROOF. We only prove the second inquality, the first one is analogous. Note that the integrand is bounded in $X \setminus B(x, \frac{R}{2})$. So for $0 < \delta \leq \frac{R}{2}$,

$$\left| \int_{R-\delta < \rho(x,y) < R} K(x,y) f(y) \, d\mu(y) \right|$$

$$\leq \frac{2^{a^3}}{\mu(B(x, \frac{R}{2}))} \sup_{y \in X} |f(x)| \cdot \mu(\{y \in X : R - \delta < \rho(x,y) < R\}).$$

By continuity from above of μ, the right factor becomes arbitrarily small as $\delta \to 0$. Thus, for small enough δ, the whole expression is $\leq \epsilon$. $\qquad\square$

LEMMA 10.1.8. *Let f be a bounded measurable function with bounded support on* X. *For all $x \in X$,*

$$T_* f(x) = \sup_{R_1 < R_2} \sup_{x' \in B(x,R_1)} \left| \int_{B(x',R_2) \setminus B(x',R_1)} K(x',y) f(y) \, d\mu(y) \right| \tag{10.1.39}$$

PROOF. We show two inequalities. Let $\epsilon > 0$. Let $R_1 < R_2$ and $x' \in B(x, R_1)$. Then for $\delta > 0$,

$$\left| \int_{R_1 < \rho(x',y) < R_2} K(x',y) f(y) \, d\mu(y) \right| \tag{10.1.40}$$

$$\leq \left| \int_{R_1 < \rho(x',y) < R_1+\delta} K(x',y) f(y) \, d\mu(y) \right| \tag{10.1.41}$$

$$+ \left| \int_{R_1+\delta \leq \rho(x',y) < R_2} K(x',y) f(y) \, d\mu(y) \right|. \tag{10.1.42}$$

By Lemma 10.1.7, we can choose δ such that (10.1.41) is bounded by ϵ. Without loss of generality, we can assume $R_1 + \delta < R_2$. Then (10.1.42) is bounded by the right hand side of (10.1.39) and we obtain

$$\leq \epsilon + \sup_{R_1 < R_2} \sup_{x' \in B(x,R_1)} \left| \int_{B(x',R_2) \setminus B(x',R_1)} K(x',y) f(y) \, d\mu(y) \right|.$$

The inequality still holds when taking the suprema over $R_1 < R_2$ and $\rho(x,x') < R_1$ in (10.1.40). Since $\epsilon > 0$ was arbitrary, this proves the first inequality.

The other direction is similar. Let $\epsilon > 0$. Let $R_1 < R_2$ and $x' \in B(x, R_1)$. Then for $\delta > 0$,

$$\left| \int_{B(x', R_2) \smallsetminus B(x', R_1)} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| \tag{10.1.43}$$

$$\leq \left| \int_{R_1 - \delta < \rho(x', y) < R_1} K(x', y) f(y) \, \mathrm{d}\mu(y) \right| \tag{10.1.44}$$

$$+ \left| \int_{R_1 - \delta < \rho(x', y) < R_2} K(x', y) f(y) \, \mathrm{d}\mu(y) \right|. \tag{10.1.45}$$

By Lemma 10.1.7, we can choose δ such that (10.1.44) is bounded by ϵ. Without loss of generality, we can assume $\rho(x, x') < R_1 - \delta$. Then (10.1.45) is bounded by the left hand side of (10.1.39) and we obtain

$$\leq \epsilon + \sup_{R_1 < R_2} \sup_{x' \in B(x, R_1)} \left| \int_{R_1 < \rho(x', y) < R_2} K(x', y) f(y) \, \mathrm{d}\mu(y) \right|.$$

The inequality still holds when taking the suprema over $R_1 < R_2$ and $\rho(x, x') < R_1$ in (10.1.40). Since $\epsilon > 0$ was arbitrary, this proves the second inequality. $\qquad \square$

PROOF OF LEMMA 10.0.2. Fix g as in the Lemma. Applying Lemma 10.1.6 with a sequence of r tending to 0 and using Lebesgue monotone convergence shows

$$\|T_0 g\|_2 \leq 2^{a^3 + 24a + 6} \|g\|_2, \tag{10.1.46}$$

where

$$T_0 g(x) := \sup_{0 < R} \sup_{x' \in B(x, R)} \left| \int_{X \smallsetminus B(x', R)} K(x', y) g(y) \, d\mu(y) \right|. \tag{10.1.47}$$

We now write using Lemma 10.1.8 and the triangle inequality,

$$T_* g(x) \leq \sup_{0 < R_1 < R_2} \sup_{x' \in B(x, R_1)} \left| \int_{X \smallsetminus B(x', R_1)} K(x', y) g(y) \, d\mu(y) \right|$$

$$+ \sup_{0 < R_1 < R_2} \sup_{x' \in B(x, R_1)} \left| \int_{X \smallsetminus B(x', R_2)} K(x', y) g(y) \, d\mu(y) \right|.$$

Noting that the first integral does not depend on R_2 and estimating the second summand by the larger supremum over all $x' \in B(x, R_2)$, at which time the integral does not depend on R_1, we estimate further

$$\leq \sup_{0 < R_1} \sup_{x' \in B(x, R_1)} \left| \int_{X \smallsetminus B(x', R_1)} K(x', y) g(y) \, d\mu(y) \right|$$

$$+ \sup_{0 < R_2} \sup_{x' \in B(x, R_2)} \left| \int_{X \smallsetminus B(x', R_2)} K(x', y) g(y) \, d\mu(y) \right|.$$

Applying the triangle inequality on the left-hand side of (10.0.5) and applying (10.1.46) twice proves (10.0.5). This completes the proof of Lemma 10.0.2. $\qquad \square$

10.2. Calderón-Zygmund Decomposition

The following lemma is Theorem 3.1(b) in [Ste93]. The proof uses Proposition 2.0.6.

LEMMA 10.2.1 (Maximal theorem). *Let $f : X \to \mathbb{C}$ be bounded, measurable, supported on a set of finite measure, and let $\alpha > 0$. Then*

$$\mu(\{x \in X : Mf(x) > \alpha\}) \leq \frac{2^{2a}}{\alpha} \int |f(y)| \, d\mu(y). \qquad (10.2.1)$$

PROOF. By definition, for each $x \in X$ with $Mf(x) > \alpha$, there exists a ball B_x such that $x \in B_x$ and

$$\alpha\mu(B_x) < \int_{B_x} |f(y)| \, d\mu(y). \qquad (10.2.2)$$

Since $\{x \in X : Mf(x) > \alpha\}$ is open and μ is inner regular on open sets, it suffices to show that

$$\mu(E) \leq \frac{2^{2a}}{\alpha} \int |f(y)| \, d\mu(y)$$

for every compact $E \subset \{x \in X : Mf(x) > \alpha\}$. For such an E, by compactness, we can select a finite subcollection $\mathcal{B} \subset \{B_x : x \in E\}$ that covers E. By (2.0.43) applied to (10.2.2),

$$\alpha\mu(\bigcup \mathcal{B}) \leq 2^{2a} \int |f(y)| \, d\mu(y) \qquad (10.2.3)$$

and hence

$$\mu(E) \leq \mu(\bigcup \mathcal{B}) \leq \frac{2^{2a}}{\alpha} \int |f(y)| \, d\mu(y).$$

$\square$

LEMMA 10.2.2 (Lebesgue differentiation). *Let f be a bounded measurable function supported on a set of finite measure. Then for μ almost every x, we have*

$$\lim_{n \to \infty} \frac{1}{\mu(B_n)} \int_{B_n} f(y) \, dy = f(x),$$

where $\{B_n\}_{n \geq 1}$ is a sequence of balls with radii $r_n > 0$ such that $x \in B_n$ for each $n \geq 1$ and

$$\lim_{n \to \infty} r_n = 0 \, .$$

PROOF. This follows from the Lebesgue differentiation theorem, which is already formalized in Lean. $\square$

LEMMA 10.2.3 (Disjoint family countable). *In a doubling metric measure space (X, ρ, μ, a), every disjoint family of balls $B_j = B(x_j, r_j)$, $j \in J$, is countable.*

PROOF. Choose an arbitrary $x \in X$ as reference point. For $q, Q \in \mathbb{Q}_+$, let $J_{q,Q}$ denote the set of all $j \in J$ such that $B_j \subset B(x, Q)$ and $r_j \geq q$. It suffices to show that all the $J_{q,Q}$ are finite. Indeed, for all $j \in J_{q,Q}$,

$$\mu(B(x, Q)) \leq \mu(B(x_j, 2Q)) = \mu(B(x_j, \frac{2Q}{r_j} r_j)) \leq 2^{a \log_2 \lceil \frac{2Q}{r_j} \rceil} \mu(B_j).$$

Since the B_j are disjoint,

$$|J_{q,Q}|\mu(B(x,Q)) \leq 2^{a\log_2\lceil\frac{2Q}{q}\rceil}\sum_{j\in J_{q,Q}}\mu(B_j) \leq 2^{a\log_2\lceil\frac{2Q}{q}\rceil}\mu(B(x,Q)) \qquad (10.2.4)$$

and hence $|J_{q,Q}| \leq 2^{a\log_2\lceil\frac{2Q}{q}\rceil}$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Calderón-Zygmund decomposition is a tool to extend L^2 bounds to L^p bounds with $p < 2$ or to the so-called weak $(1,1)$ type endpoint bound. It is classical and can be found in [Ste93].

The following lemma corresponds to Lemma 3.2 in [Ste93] with additional proof of the bounded intersection property taken from the proof of Proposition 7.1 .

LEMMA 10.2.4 (Ball covering). *Given an open set $O \neq X$, there exists a countable family of balls $B_j = B(x_j, r_j)$ such that*

$$B_j \cap B_{j'} = \varnothing \quad \text{for } j \neq j' \qquad (10.2.5)$$

and for $B_j^* := B(x_j, 3r_j)$,

$$\bigcup_j B_j^* = O \qquad (10.2.6)$$

and for $B_j^{**} := B(x_j, 7r_j)$,

$$B_j^{**} \cap (X \smallsetminus O) \neq \varnothing \quad \text{for all } j \qquad (10.2.7)$$

and we have the bounded intersection property that each $x \in O$ is contained in at most 2^{6a} of the B_j^*.

PROOF. Define for $x \in O$,

$$\delta(x) := \sup\{\delta \in \mathbb{R} : B(x,\delta) \subset O\}. \qquad (10.2.8)$$

Since O is open, and $O \neq X$, we have

$$0 < \delta(x) < \infty. \qquad (10.2.9)$$

Using Zorn's Lemma, we select a maximal disjoint subfamily of $\{B(x, \frac{\delta(x)}{6}) : x \in O\}$. We obtain a (by Lemma 10.2.3 countable) family of balls $B_j = B(x_j, \frac{\delta(x_j)}{6}), j \in J$ such that (10.2.5). (10.2.7) and $\bigcup_j B_j^* \subset O$ are also immediate. For the other inclusion, first observe that for $x, y \in X$, if $B(x, \frac{\delta(x)}{6}) \cap B(y, \frac{\delta(y)}{6}) \neq \varnothing$, then

$$\delta(x) \leq \rho(x,y) + \delta(y) \leq (\frac{\delta(x)}{6} + \frac{\delta(y)}{6}) + \delta(y) = \frac{\delta(x)}{6} + \frac{7\delta(y)}{6},$$

so

$$\delta(x) \leq 2\delta(y). \qquad (10.2.10)$$

Now let $z \in X$. By maximality, there exists some $j \in J$ with $B(z, \frac{\delta(z)}{6}) \cap B_j \neq \varnothing$. By (10.2.10),

$$\rho(z, x_j) \leq \frac{\delta(z)}{6} + \frac{\delta(x_j)}{6} \leq \frac{3\delta(x_j)}{6} = 3r_j$$

and thus $z \in B_j^*$.

We now turn to the bounded intersection property. Assume that for some $j_1, \ldots, j_N$,

$$z \in \bigcap_{k=1}^{N} B_{j_k}^*. \tag{10.2.11}$$

Similarly as above, observe for $1 \le k \le N$,

$$\delta(z) \le \rho(z, x_{j_k}) + \delta(x_{j_k}) \le \frac{\delta(x_{j_k})}{2} + \delta(x_{j_k}) = \frac{3\delta(x_{j_k})}{2} \tag{10.2.12}$$

and

$$\delta(x_{j_k}) \le \rho(x_{j_k}, z) + \delta(z) \le \frac{\delta(x_{j_k})}{2} + \delta(z),$$

so

$$\delta(x_{j_k}) \le 2\delta(z). \tag{10.2.13}$$

By (10.2.12) and (10.2.13), for all $1 \le k \le N$, $B(z, \frac{\delta(z)}{6}) \subset B(x_{j_k}, 5r_{j_k})$ and $B_{j_k} \subset B(z, \frac{8\delta(z)}{6})$. Using this and (10.2.5), we obtain

$$N\mu(B(z, \frac{\delta(z)}{6})) \le \sum_{k=1}^{N} \mu(B(x_{j_k}, 5r_j)) \le 2^{3a} \sum_{k=1}^{N} \mu(B_{j_k}) \tag{10.2.14}$$

$$= 2^{3a}\mu(\bigcup_{k=1}^{N} B_{j_k}) \le 2^{3a}\mu(B(z, \frac{8\delta(z)}{6})) \le 2^{6a}\mu(B(z, \frac{\delta(z)}{6})) \tag{10.2.15}$$

and conclude $N \le 2^{6a}$. $\qquad\qquad\square$

Most of the next lemma and its proof is taken from Theorem 4.2 in [Ste93].

LEMMA 10.2.5 (Calderon Zygmund decomposition). *Let f be a bounded, measurable function supported on a set of finite measure and let $\alpha > \frac{1}{\mu(X)} \int |f| \, d\mu$. Then there exists a measurable function g, a countable family of balls B_j^* (where we allow $B_1^* = X$ in the special case that $\mu(X) < \infty$) such that each $x \in X$ is contained in at most 2^{6a} of the B_j^*, and a countable family of measurable functions $\{b_j\}_{j \in J}$ such that for all $x \in X$*

$$f(x) = g(x) + \sum_j b_j(x) \tag{10.2.16}$$

and such that the following holds. For almost every $x \in X$,

$$|g(x)| \le 2^{3a}\alpha. \tag{10.2.17}$$

We have

$$\int |g(y)| \, d\mu(y) \le \int |f(y)| \, d\mu(y). \tag{10.2.18}$$

For every j

$$\operatorname{supp} b_j \subset B_j^*. \tag{10.2.19}$$

For every j

$$\int_{B_j^*} b_j(x) \, d\mu(x) = 0, \tag{10.2.20}$$

and

$$\int_{B_j^*} |b_j(x)| \, d\mu(x) \le 2^{2a+1} \alpha \mu(B_j^*). \tag{10.2.21}$$

We have

$$\sum_j \mu(B_j^*) \le \frac{2^{4a}}{\alpha} \int |f(y)| \, d\mu(y) \tag{10.2.22}$$

and

$$\sum_j \int_{B_j^*} |b_j(y)| \, d\mu(y) \le 2 \int |f(y)| \, d\mu(y). \tag{10.2.23}$$

PROOF. Let $E_\alpha := \{x \in X : Mf(x) > \alpha\}$. Then E_α is open. Assume first that $E_\alpha \ne X$. We apply Lemma 10.2.4 with $O = E_\alpha$ to obtain the family $B_j, j \in J,$. Without loss of generality, we can assume $J = \mathbb{N}$. Define

$$Q_j := B_j^* \smallsetminus \left(\bigcup_{i<j} Q_i \cup \bigcup_{i>j} B_i \right). \tag{10.2.24}$$

Then the $B_j \subset Q_j \subset B_j^*$, the Q_j are pairwise disjoint and $\bigcup_j Q_j = E_\alpha$. Define

$$g(x) := \begin{cases} f(x), & x \in X \smallsetminus E_\alpha, \\ \frac{1}{\mu(Q_j)} \int_{Q_j} f(y) \, d\mu(y), & x \in Q_j, \end{cases} \tag{10.2.25}$$

and, for each j,

$$b_j(x) := \mathbf{1}_{Q_j}(x) \left(f(x) - \frac{1}{\mu(Q_j)} \int_{Q_j} f(y) \, d\mu(y) \right). \tag{10.2.26}$$

Then (10.2.16), (10.2.19) and (10.2.20) are true by construction. For (10.2.17), we first do the case $x \in X \smallsetminus E_\alpha$. By definition of Mf,

$$\frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y) \le \alpha \tag{10.2.27}$$

for every ball $B \subset X$ with $x \in B$. It follows by Lemma 10.2.2 that for almost every $x \in X \smallsetminus E_\alpha$, $|f(x)| \le \alpha$. In the case $x \in E_\alpha$, there exists some $j \in J$ with $x \in Q_j$ and we have that

$$\frac{1}{\mu(B_j^{**})} \int_{B_j^{**}} |f(y)| \, d\mu(y) \le \alpha \tag{10.2.28}$$

because $B_j^{**} \cap (X \smallsetminus E_\alpha) \ne \varnothing$. We get

$$|g(x)| \le \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)| \, d\mu(y) \le \frac{1}{\mu(B_j)} \int_{B_j^{**}} |f(y)| \, d\mu(y) \le 2^{3a} \alpha. \tag{10.2.29}$$

To prove (10.2.18), we estimate

$$\int |g(z)| \, d\mu(z) \le \int_{X \smallsetminus E_\alpha} |f(z)| \, d\mu(z) + \sum_j \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)| \, d\mu(y) \, d\mu(z)$$

$$= \int |f(z)| \, d\mu(z).$$

Using the triangle inequality, we have that

$$\int_{B_j^*} |b_j(y)|\, dy \le \int_{Q_j} |f(y)|\, d\mu(y) + \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)|\, d\mu(x)\, d\mu(y) \qquad (10.2.30)$$

$$= 2\int_{Q_j} |f(y)|\, dy. \qquad (10.2.31)$$

With (10.2.28), we estimate further

$$\le 2\int_{B_j^{**}} |f(y)|\, dy \le 2\mu(B_j^{**})\alpha \le 2^{2a+1}\alpha\mu(B_j^*) \qquad (10.2.32)$$

to obtain (10.2.21). Further, summing up (10.2.30) in j yields (10.2.23). At last, we estimate with Lemma 10.2.1

$$\sum_j \mu(B_j^*) \le 2^{2a}\sum_j \mu(B_j) \le 2^{2a}\mu(E_\alpha) \le \frac{2^{4a}}{\alpha}\int |f(y)|\, d\mu(y), \qquad (10.2.33)$$

proving (10.2.22).

Assume now that $E_\alpha = X$. It follows from Lemma 10.2.1 that then $\mu(X) < \infty$. Define

$$g := \frac{1}{\mu(X)}\int |f(y)|\, d\mu(y)$$

and

$$b_1 := f - g.$$

Then $f = g + b_1$ and $\operatorname{supp} b_1 \subset B_1^* := X$ and (10.2.16), (10.2.18), (10.2.19), (10.2.20) all hold immediately. By assumption, $\alpha > \frac{1}{\mu(X)}\int |f|\, d\mu = g$, so (10.2.17) holds. We also have, using the definitions and the same assumption,

$$\int |b_1(y)|\, d\mu(y) \le 2\int |f(y)|\, d\mu(y) \le 2\alpha\mu(X), \qquad (10.2.34)$$

which verifies both (10.2.23) and (10.2.21). Finally, by Lemma 10.2.1,

$$\mu(X) \le \frac{2^{2a}}{\alpha}\int |f(y)|\, d\mu(y),$$

which shows (10.2.22). $\qquad\qquad\square$

We use Lemma 10.2.5 to prove Lemma 10.0.3. For the remainder of this section, let $f : X \to \mathbb{C}$, $r > 0$ and $\alpha > 0$ as in the lemma. We define the constant

$$c := 2^{-a^3 - 12a - 4} \qquad (10.2.35)$$

and $\alpha' := c\alpha$. If $\alpha' \le \frac{1}{\mu(X)}\int |f|\, d\mu$, then we directly have

$$\mu\left(\{x \in X : |T_r f(x)| > \alpha\}\right) \le \mu(X) \le \frac{1}{\alpha'}\int |f(y)|\, d\mu(y) \le \frac{2^{a^3+19a}}{\alpha}\int |f(y)|\, d\mu(y),$$

which proves (10.0.9) So assume from now on that $\alpha' > \frac{1}{\mu(X)}\int |f|\, d\mu$. Using Lemma 10.2.5 for f and α', we obtain the decomposition

$$f = g + b = g + \sum_j b_j$$

such that the properties (10.2.16)-(10.2.23) are satisfied (with α' replacing α). We rename B_j^* to B_j and let

$$B_j = B(x_j, r_j). \tag{10.2.36}$$

Define

$$B_j' := B(x_j, 2r_j). \tag{10.2.37}$$

(In the special case $B_j = X$, we define $B_j' := X$.) Then B_j' is a ball with the same center as B_j but with

$$\mu(B_j') \le 2^a \mu(B_j). \tag{10.2.38}$$

Let

$$\Omega := \bigcup_j B_j'. \tag{10.2.39}$$

We deal with $T_r g$ and $T_r b$ separately in the following lemmas.

LEMMA 10.2.6 (Estimate good).

$$\mu\left(\{x \in X : |T_r g(x)| > \alpha/2\}\right) \le \frac{2^{2a^3+3a+2}c}{\alpha} \int |f(y)| \, d\mu(y).$$

PROOF. We estimate using monotonicity of the integral

$$\mu\left(\{x \in X : |T_r g(x)| > \alpha/2\}\right) \le \frac{4}{\alpha^2} \int |T_r g(y)|^2 \, d\mu(y).$$

Using (10.0.8) followed by (10.2.17) and (10.2.18), we estimate the right hand side above by

$$\le \frac{4 \cdot 2^{2a^3}}{\alpha^2} \int |g(y)|^2 \, d\mu(y) \le \frac{2^{2a^3+3a+2}c}{\alpha} \int |g(y)| \, dy \le \frac{2^{2a^3+3a+2}c}{\alpha} \int |f(y)| \, d\mu(y). \tag{10.2.40}$$

$\square$

LEMMA 10.2.7. *Let $x \in X \setminus \Omega$. Then*

$$|T_r b(x)| \le 3F(x) + \alpha/8,$$

where

$$F(x) := 2^{a^3+2a+1} c\alpha \sum_{j \in J} \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{\mu(B_j)}{V(x, x_j)}.$$

PROOF. We decompose the index set J into the following disjoint sets:

$$\begin{aligned}
\mathcal{J}_1(x) &:= \{j : r + r_j \le \rho(x, x_j)\}, \\
\mathcal{J}_2(x) &:= \{j : r - r_j \le \rho(x, x_j) < r + r_j\}, \\
\mathcal{J}_3(x) &:= \{j : \rho(x, x_j) < r - r_j\}.
\end{aligned}$$

Then

$$|T_r b(x)| \leq \sum_{j \in \mathcal{J}_1(x)} |T_r b_j(x)| \tag{10.2.41}$$

$$+ \sum_{j \in \mathcal{J}_2(x)} |T_r b_j(x)| \tag{10.2.42}$$

$$+ \sum_{j \in \mathcal{J}_3(x)} |T_r b_j(x)|. \tag{10.2.43}$$

For all $j \in \mathcal{J}_3(x)$, $\operatorname{supp} b_j \subset B_j \subset B(x, r)$, and thus $T_r b_j(x) = 0$, so $(10.2.43) = 0$.

Next, for $j \in \mathcal{J}_1(x)$, $\operatorname{supp} b_j \subset B_j \subset X \smallsetminus B(x, r)$, and we have

$$T_r b_j(x) = \int_{X \smallsetminus B(x,r)} K(x, y) b_j(y) \, d\mu(y) = \int_{B_j} K(x, y) b_j(y) \, d\mu(y).$$

Using (10.2.20), the above is equal to

$$\int_{B_j} (K(x, y) - K(x, x_j)) b_j(y) \, d\mu(y).$$

Since $x \in X \smallsetminus \Omega$, we have for each $y \in B_j$ that

$$\rho(x, x_j) \geq 2r_j > 2\rho(x_j, y), \tag{10.2.44}$$

so we can apply (1.0.15) to estimate

$$(10.2.41) \leq \sum_{j \in \mathcal{J}_1(x)} \int_{B_j} \left(\frac{\rho(x_j, y)}{\rho(x, x_j)} \right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x, x_j)} |b_j(y)| \, d\mu(y)$$

$$\leq 2^{a^3} \sum_j \left(\frac{r_j}{\rho(x, x_j)} \right)^{\frac{1}{a}} \frac{1}{V(x, x_j)} \int_{B_j} |b_j(y)| \, dy \tag{10.2.45}$$

and by (10.2.21),

$$\leq 2^{a^3 + 2a + 1} c\alpha \sum_j \left(\frac{r_j}{\rho(x, x_j)} \right)^{\frac{1}{a}} \frac{\mu(B_j)}{V(x, x_j)} = F(x). \tag{10.2.46}$$

Next, we estimate (10.2.42). For each $j \in \mathcal{J}_2(x)$, set

$$d_j := \frac{1}{\mu(B_j)} \int_{B_j} \mathbf{1}_{X \smallsetminus B(x,r)}(y) b_j(y) \, dy.$$

Then by (10.2.21)

$$|d_j| \leq 2^{2a+1} c\alpha. \tag{10.2.47}$$

For each $j \in \mathcal{J}_2(x)$, we have

$$T_r b_j(x) = \int_{B_j} K(x, y) (\mathbf{1}_{X \smallsetminus B(x,r)}(y) b_j(y) - d_j) \, dy + \int_{B_j} d_j K(x, y) \, dy$$

$$= \int_{B_j} (K(x, y) - K(x, x_j)) (\mathbf{1}_{X \smallsetminus B(x,r)}(y) b_j(y) - d_j) \, dy + \int_{B_j} d_j K(x, y) \, dy.$$

Thus, using the triangle inequality, the estimate above and (10.2.47), we obtain

$$|T_r b_j(x)| \leq$$

$$\int_{B_j} |K(x,y) - K(x,x_j)| \left(|b_j(y)| + 2^{2a+1}c\alpha\right) dy + 2^{2a+1}c\alpha \int_{B_j} |K(x,y)| \, dy. \quad (10.2.48)$$

By (10.2.44), we can apply (1.0.15) and arguing as in (10.2.46), we get that

$$(10.2.42) \leq 2F(x) + 2^{2a+1}c\alpha \sum_{j \in \mathcal{J}_2(x)} \int_{B_j} |K(x,y)| \, d\mu(y), \quad (10.2.49)$$

with F as in (10.2.46). Define

$$A := \bigcup_{j \in \mathcal{J}_2(x)} B_j.$$

We claim that

$$A \subset B(x, 3r) \setminus B(x, \frac{r}{3}). \quad (10.2.50)$$

Indeed, for each $j \in \mathcal{J}_2(x)$ and $y \in B_j$,

$$\rho(x, x_j) < r + r_j \leq r + \frac{1}{2}\rho(x, x_j) \implies \rho(x, x_j) < 2r$$

and hence

$$\rho(x, y) \leq \rho(x, x_j) + \rho(x_j, y) < 2r + r_j \leq 2r + \frac{1}{2}\rho(x, x_j) < 3r.$$

For the lower bound, we observe

$$\rho(x, x_j) \geq r - r_j \geq r - \frac{1}{2}\rho(x, x_j) \implies \rho(x, x_j) \geq \frac{2}{3}r,$$

and conclude

$$\rho(x, y) \geq \rho(x, x_j) - \rho(y, x_j) \geq \rho(x, x_j) - r_j \geq \rho(x, x_j) - \frac{1}{2}\rho(x, x_j) \geq \frac{1}{3}r.$$

Using the bounded intersection property of the B_j, (10.2.50) and (1.0.14), we get

$$\sum_{j \in \mathcal{J}_2(x)} \int_{B_j} |K(x,y)| \, d\mu(y) \leq 2^{6a} \int_A |K(x,y)| \, d\mu(y) \quad (10.2.51)$$

$$\leq 2^{6a} \int_{B(x,3r) \setminus B(x, \frac{r}{3})} |K(x,y)| \, d\mu(y) \quad (10.2.52)$$

$$\leq 2^{6a} \int_{B(x,3r) \setminus B(x, \frac{r}{3})} \frac{2^{a^3}}{V(x,y)} \, d\mu(y) \quad (10.2.53)$$

$$\leq 2^{a^3+6a} \int_{B(x,3r) \setminus B(x, \frac{r}{3})} \frac{1}{\mu(B(x, \frac{r}{3}))} \, d\mu(y) \quad (10.2.54)$$

$$\leq 2^{a^3+6a} \frac{\mu(B(x,3r))}{\mu(B(x, \frac{r}{3}))} \quad (10.2.55)$$

$$\leq 2^{a^3+10a}. \quad (10.2.56)$$

Combining the estimates (10.2.46) for (10.2.41), (10.2.49) for (10.2.42), and (10.2.56), we get

$$|T_r b(x)| \leq 3F(x) + 2^{a^3+12a+1} c\alpha.$$

By the definition (10.2.35) of c, this equals

$$3F(x) + \alpha/8.$$

$\square$

LEMMA 10.2.8. *For F as defined in Lemma 10.2.7, we have*

$$\mu(\{x \in X \smallsetminus \Omega : F(x) > \alpha/8\}) \leq \frac{2^{a^3+9a+4}}{\alpha} \int |f(y)|\, d\mu(y)\,.$$

PROOF. We estimate

$$\mu(\{x \in X \smallsetminus \Omega : F(x) > \alpha/8\}) \leq \frac{8}{\alpha} \int_{X\smallsetminus\Omega} F(x)\, d\mu(x) \tag{10.2.57}$$

$$\leq \frac{8}{\alpha} \int_{X\smallsetminus\Omega} 2^{a^3+2a+1} c\alpha \sum_j \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{\mu(B_j)}{V(x, x_j)}\, d\mu(x) \tag{10.2.58}$$

$$\leq 2^{a^3+2a+4} c \sum_j \mu(B_j) \int_{X\smallsetminus B_j'} \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{1}{V(x, x_j)}\, d\mu(x) \tag{10.2.59}$$

Using

$$V(x, x_j) = \mu(B(x, \rho(x, x_j))) \geq 2^{-a} \mu(B(x, 2\rho(x, x_j))) \geq 2^{-a} \mu(B(x_j, \rho(x_j, x))),$$

we have for all $j \in J$,

$$\int_{X\smallsetminus B_j'} \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{1}{V(x, x_j)}\, d\mu(x)$$

$$\leq 2^a \int_{X\smallsetminus B_j'} \left(\frac{r_j}{\rho(x, x_j)}\right)^{\frac{1}{a}} \frac{1}{\mu(B(x_j, \rho(x_j, x)))}\, d\mu(x)$$

$$\leq 2^a \sum_{n=1}^{\infty} \int_{B(x_j, 2^{n+1}r_j)\smallsetminus B(x_j, 2^n r_j)} \left(\frac{r_j}{2^n r_j}\right)^{\frac{1}{a}} \frac{1}{\mu(B(x_j, 2^n r_j))}\, d\mu(x)$$

$$\leq 2^a \sum_{n=1}^{\infty} 2^{-\frac{n}{a}} \frac{\mu(B(x_j, 2^{n+1}r_j))}{\mu(B(x_j, 2^n r_j))}$$

$$\leq 2^{3a},$$

where we used Lemma 10.1.1 in the last step. Plugging this into (10.2.59) and using (10.2.22), we conclude that

$$\mu(\{x \in X \smallsetminus \Omega : F(x) > \alpha/8\}) \leq \frac{2^{a^3+9a+4}}{\alpha} \int |f(y)|\, d\mu(y)\,.$$

$\square$

LEMMA 10.2.9 (Estimate bad). *We have*

$$\mu\left(\{x \in X : |T_r b(x)| > \alpha/2\}\right) \leq \frac{\frac{2^{5a}}{c} + 2^{a^3+9a+4}}{\alpha} \int |f(y)| \, d\mu(y).$$

PROOF. We estimate

$$\mu\left(\{x \in X : |T_r b(x)| > \alpha/2\}\right)$$
$$\leq \mu(\Omega) + \mu\left(\{x \in X \smallsetminus \Omega : |T_r b(x)| > \alpha/2\}\right). \qquad (10.2.60)$$

Using (10.2.38) and (10.2.22), we conclude that

$$\mu(\Omega) \leq \sum_j \mu(B'_j) \leq 2^a \sum_j \mu(B_j) \leq \frac{2^{5a}}{c\alpha} \int |f(y)| \, d\mu(y). \qquad (10.2.61)$$

It follows from Lemma 10.2.7 and the triangle inequality that

$$\mu(\{x \in X \smallsetminus \Omega : |T_r b(x)| > \alpha/2\}) \leq \mu(\{x \in X \smallsetminus \Omega : F(x) > \alpha/8\}). \qquad (10.2.62)$$

The claim now follows from (10.2.60), (10.2.62) and (10.2.8). □

PROOF OF LEMMA 10.0.3. It follows by the triangle inequality and subadditivity of μ that

$$\mu\left(\{x \in X : |T_r f(x)| > \alpha\}\right)$$
$$\leq \mu\left(\{x \in X : |T_r g(x)| > \alpha/2\}\right) + \mu\left(\{x \in X : |T_r b(x)| > \alpha/2\}\right).$$

Using Lemma 10.2.6, Lemma 10.2.9 and the definition (10.2.35) of c, we get

$$\leq \frac{2^{2a^3+3a+2}c + \frac{2^{5a}}{c} + 2^{a^3+9a+4}}{\alpha} \int |f(y)| \, d\mu(y)$$

$$= \frac{2^{a^3-9a-2} + 2^{a^3+17a+4} + 2^{a^3+9a+4}}{\alpha} \int |f(y)| \, d\mu(y)$$

$$\leq \frac{2^{a^3+19a}}{\alpha} \int |f(y)| \, d\mu(y).$$

This proves (10.0.9). □

CHAPTER 11

Proof of The Classical Carleson Theorem

The convergence of partial Fourier sums is proved in Section 11.1 in two steps. In the first step, we establish convergence on a suitable dense subclass of functions. We choose smooth functions as subclass, the convergence is stated in Lemma 11.1.2 and proved in Section 11.2. In the second step, one controls the relevant error of approximating a general function by a function in the subclass. This is stated in Lemma 11.1.3 and proved in Section 11.6. The proof relies on a bound on the real Carleson maximal operator stated in Lemma 11.1.4 and proved in Section 11.7, which involves showing that the real line fits into the setting of Chapter 2. This latter proof refers to the two-sided variant of the Carleson Theorem 10.0.1. Two assumptions in Theorem 1.0.2 require more work. The boundedness of the operator T_r defined in (10.0.2) is established in 11.1.5. This lemma is proved in Section 11.3. The cancellative property is verified by Lemma 11.1.6, which is proved in Section 11.4. Several further auxiliary lemmas are stated and proved in Section 11.1, the proof of one of these auxiliary lemmas, Lemma 11.1.10, is done in Section 11.5.

All subsections past Section 11.1 are mutually independent.

11.1. The classical Carleson theorem

Let a uniformly continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ and $\epsilon > 0$ be given. Let

$$C_{a,q} := \frac{2^{452a^3}}{(q-1)^6} \tag{11.1.1}$$

denote the constant from Theorem 10.0.1. Define

$$\epsilon' := \frac{\epsilon}{4C_\epsilon}, \tag{11.1.2}$$

where

$$C_\epsilon = \left(\frac{8}{\pi\epsilon}\right)^{\frac{1}{2}} C_{4,2} + \pi.$$

Since f is continuous and periodic, f is uniformly continuous. Thus, there is a $0 < \delta < \pi$ such that for all $x, x' \in \mathbb{R}$ with $|x - x'| \leq \delta$ we have

$$|f(x) - f(x')| \leq \epsilon'. \tag{11.1.3}$$

Define

$$f_0 := f * \phi_\delta, \tag{11.1.4}$$

where ϕ_δ is a nonnegative smooth bump function with $\mathrm{supp}(\phi_\delta) \subset (-\delta, \delta)$ and $\int_{\mathbb{R}} \phi_\delta(x)\,dx = 1$.

LEMMA 11.1.1 (smooth approximation). *The function f_0 is 2π-periodic. The function f_0 is smooth (and therefore measurable). The function f_0 satisfies for all $x \in \mathbb{R}$:*

$$|f(x) - f_0(x)| \leq \epsilon', \tag{11.1.5}$$

PROOF. Most of this is part of the Lean library. □

We prove in Section 11.2:

LEMMA 11.1.2 (convergence for smooth). *There exists some $N_0 \in \mathbb{N}$ such that for all $N > N_0$ and $x \in [0, 2\pi]$ we have*

$$|S_N f_0(x) - f_0(x)| \leq \frac{\epsilon}{4}. \tag{11.1.6}$$

We prove in Section 11.6:

LEMMA 11.1.3 (control approximation effect). *There is a set $E \subset \mathbb{R}$ with Lebesgue measure $|E| \leq \epsilon$ such that for all*

$$x \in [0, 2\pi) \smallsetminus E \tag{11.1.7}$$

we have

$$\sup_{N \geq 0} |S_N f(x) - S_N f_0(x)| \leq \frac{\epsilon}{4}. \tag{11.1.8}$$

We are now ready to prove classical Carleson:

PROOF OF THEOREM 1.0.1. Let N_0 be as in Lemma 11.1.2. For every

$$x \in [0, 2\pi) \smallsetminus E, \tag{11.1.9}$$

and every $N > N_0$ we have by the triangle inequality

$$|f(x) - S_N f(x)|$$

$$\leq |f(x) - f_0(x)| + |f_0(x) - S_N f_0(x)| + |S_N f_0(x) - S_N f(x)|. \tag{11.1.10}$$

Using Lemmas 11.1.1 to 11.1.3, we estimate (11.1.10) by

$$\leq \epsilon' + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \epsilon. \tag{11.1.11}$$

This shows (1.0.3) for the given E and N_0. □

Let $\kappa : \mathbb{R} \to \mathbb{R}$ be the function defined by $\kappa(0) = 0$ and for $0 < |x| < 1$

$$\kappa(x) = \frac{1 - |x|}{1 - e^{ix}} \tag{11.1.12}$$

and for $|x| \geq 1$,

$$\kappa(x) = 0. \tag{11.1.13}$$

Note that this function is continuous at every point x with $|x| > 0$.

The proof of Lemma 11.1.3 will use the following Lemma 11.1.4, which itself is proven in Section 11.7 as an application of Theorem 1.0.2.

LEMMA 11.1.4 (real Carleson). *Let F, G be Borel subsets of $\mathbb{R}$ with finite measure. Let f be a bounded measurable function on $\mathbb{R}$ with $|f| \leq \mathbf{1}_F$. Then*

$$\left| \int_G Tf(x)\,dx \right| \leq C_{4,2} |F|^{\frac{1}{2}} |G|^{\frac{1}{2}}, \tag{11.1.14}$$

where

$$Tf(x) = \sup_{n \in \mathbb{Z}} \sup_{r > 0} \left| \int_{r < |x-y| < 1} f(y)\kappa(x-y)e^{iny}\,dy \right|. \tag{11.1.15}$$

One of the main assumptions of Theorem 10.0.1, concerning the operator T_r defined in (10.0.2), is verified by the following lemma, which is proved in Section 11.3.

LEMMA 11.1.5 (Hilbert strong 2 2). *Let $0 < r < 1$. Let f be a bounded, measurable function on $\mathbb{R}$ with bounded support. Then*

$$\|H_r f\|_2 \leq 2^{13} \|f\|_2, \tag{11.1.16}$$

where

$$H_r f(x) := T_r f(x) = \int_{r \leq \rho(x,y)} \kappa(x-y)f(y)\,dy \tag{11.1.17}$$

The next lemma will be used to verify that the collection Θ of modulation functions in our application of Theorem 1.0.2 satisfies the condition (1.0.12). It is proved in Section 11.4.

LEMMA 11.1.6 (van der Corput). *Let $\alpha \leq \beta$ be real numbers. Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable function and assume*

$$\|g\|_{Lip(\alpha,\beta)} := \sup_{\alpha \leq x \leq \beta} |g(x)| + \frac{|\beta - \alpha|}{2} \sup_{\alpha \leq x < y \leq \beta} \frac{|g(y) - g(x)|}{|y - x|} < \infty. \tag{11.1.18}$$

Then for any $\alpha \leq \beta$ and $n \in \mathbb{Z}$ we have

$$\int_\alpha^\beta g(x)e^{inx}\,dx \leq 2\pi |\beta - \alpha| \|g\|_{Lip(\alpha,\beta)} (1 + |n||\beta - \alpha|)^{-1}. \tag{11.1.19}$$

We close this section with six lemmas that are used across the following subsections.

LEMMA 11.1.7 (mean zero oscillation). *Let $n \in \mathbb{Z}$ with $n \neq 0$, then*

$$\int_0^{2\pi} e^{inx}\,dx = 0. \tag{11.1.20}$$

PROOF. We have

$$\int_0^{2\pi} e^{inx}\,dx = \left[\frac{1}{in} e^{inx} \right]_0^{2\pi} = \frac{1}{in}(e^{2\pi in} - e^{2\pi i0}) = \frac{1}{in}(1 - 1) = 0. \qquad \square$$

LEMMA 11.1.8 (Dirichlet kernel). *We have for every 2π-periodic bounded measurable f and every $N \geq 0$*

$$S_N f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) K_N(x - y) \, dy \qquad (11.1.21)$$

where K_N is the 2π-periodic continuous function of $\mathbb{R}$ given by

$$\sum_{n=-N}^{N} e^{inx'} . \qquad (11.1.22)$$

We have for $e^{ix'} \neq 1$ that

$$K_N(x') = \frac{e^{iNx'}}{1 - e^{-ix'}} + \frac{e^{-iNx'}}{1 - e^{ix'}} . \qquad (11.1.23)$$

PROOF. We have by definitions and interchanging sum and integral

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

$$= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{in(x-y)} \, dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(y) \sum_{n=-N}^{N} e^{in(x-y)} \, dy . \qquad (11.1.24)$$

This proves the first statement of the lemma. By a telescoping sum, we have for every $x' \in \mathbb{R}$

$$\left(e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'} \right) \sum_{n=-N}^{N} e^{inx'} = e^{(N+\frac{1}{2})ix'} - e^{-(N+\frac{1}{2})ix'} . \qquad (11.1.25)$$

If $e^{ix'} \neq 1$, the first factor on the left-hand side is not 0 and we may divide by this factor to obtain

$$\sum_{n=-N}^{N} e^{inx'} = \frac{e^{i(N+\frac{1}{2})x'}}{e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}} - \frac{e^{-i(N+\frac{1}{2})x'}}{e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}} = \frac{e^{iNx'}}{1 - e^{-ix'}} + \frac{e^{-iNx'}}{1 - e^{ix'}} . \qquad (11.1.26)$$

This proves the second part of the lemma. □

LEMMA 11.1.9 (lower secant bound). *Let $\eta > 0$ and $-2\pi + \eta \leq x \leq 2\pi - \eta$ with $|x| \geq \eta$. Then*

$$|1 - e^{ix}| \geq \frac{2}{\pi} \eta \qquad (11.1.27)$$

PROOF. We have

$$|1 - e^{ix}| = \sqrt{(1 - \cos(x))^2 + \sin^2(x)} \geq |\sin(x)| .$$

If $0 \leq x \leq \frac{\pi}{2}$, then we have from concavity of $\sin$ on $[0, \pi]$ and $\sin(0) = 0$ and $\sin(\frac{\pi}{2}) = 1$

$$|\sin(x)| \geq \frac{2}{\pi} x \geq \frac{2}{\pi} \eta .$$

When $x \in \frac{m\pi}{2} + [0, \frac{\pi}{2}]$ for $m \in \{-4, -3, -2, -1, 1, 2, 3\}$ one can argue similarly. $\square$

The following lemma will be proved in Section 11.5.

LEMMA 11.1.10 (spectral projection bound). *Let f be a bounded 2π-periodic measurable function. Then, for all $N \geq 0$*

$$\|S_N f\|_{L^2[-\pi,\pi]} \leq \|f\|_{L^2[-\pi,\pi]}. \tag{11.1.28}$$

LEMMA 11.1.11 (Hilbert kernel bound). *For $x, y \in \mathbb{R}$ with $x \neq y$ we have*

$$|\kappa(x-y)| \leq 2^2 (2|x-y|)^{-1}. \tag{11.1.29}$$

PROOF. Fix $x \neq y$. If $\kappa(x-y)$ is zero, then (11.1.29) is evident. Assume $\kappa(x-y)$ is not zero, then $0 < |x-y| < 1$. We have

$$|\kappa(x-y)| = \left| \frac{1 - |x-y|}{1 - e^{i(x-y)}} \right|. \tag{11.1.30}$$

We estimate with Lemma 11.1.9

$$|\kappa(x-y)| \leq \frac{1}{|1 - e^{i(x-y)}|} \leq \frac{2}{|x-y|}. \tag{11.1.31}$$

This proves (11.1.29) in the given case and completes the proof of the lemma. $\square$

LEMMA 11.1.12 (Hilbert kernel regularity). *For $x, y, y' \in \mathbb{R}$ with $x \neq y, y'$ and*

$$2|y - y'| \leq |x - y|, \tag{11.1.32}$$

we have

$$|\kappa(x-y) - \kappa(x-y')| \leq 2^8 \frac{1}{|x-y|} \frac{|y-y'|}{|x-y|}. \tag{11.1.33}$$

PROOF. Upon replacing y by $y - x$ and y' by $y' - x$ on the left-hand side of (11.1.32), we can assume that $x = 0$. Then the assumption (11.1.32) implies that y and y' have the same sign. Since $\kappa(y) = \bar{\kappa}(-y)$ we can assume that they are both positive. Then it follows from (11.1.32) that

$$\frac{y}{2} \leq y'.$$

We distinguish four cases. If $y, y' \leq 1$, then we have

$$|\kappa(-y) - \kappa(-y')| = \left| \frac{1-y}{1 - e^{-iy}} - \frac{1-y'}{1 - e^{-iy'}} \right|$$

and by the fundamental theorem of calculus

$$= \left| \int_{y'}^{y} \frac{-1 + e^{-it} + i(1-t)e^{it}}{(1 - e^{-it})^2} \, dt \right|.$$

Using $y' \geq \frac{y}{2}$ and Lemma 11.1.9, we bound this by

$$\leq |y - y'| \sup_{\frac{y}{2} \leq t \leq 1} \frac{3}{|1 - e^{-it}|^2} \leq 3|y - y'|(2\frac{2}{y})^2 \leq 2^6 \frac{|y - y'|}{|y|^2}.$$

If $y \leq 1$ and $y' > 1$, then $\kappa(-y') = 0$ and we have from the first case

$$|\kappa(-y) - \kappa(-y')| = |\kappa(-y) - \kappa(-1)| \leq 2^6 \frac{|y - 1|}{|y|^2} \leq 2^6 \frac{|y - y'|}{|y|^2} \,.$$

Similarly, if $y > 1$ and $y' \leq 1$, then $\kappa(-y) = 0$ and we have from the first case

$$|\kappa(-y) - \kappa(-y')| = |\kappa(-y') - \kappa(-1)| \leq 2^6 \frac{|y' - 1|}{|y'|^2} \leq 2^6 \frac{|y - y'|}{|y'|^2} \,.$$

Using again $y' \geq \frac{y}{2}$, we bound this by

$$\leq 2^6 \frac{|y - y'|}{|y/2|^2} = 2^8 \frac{|y - y'|}{|y|^2}$$

Finally, if $y, y' > 1$ then

$$|\kappa(-y) - \kappa(-y')| = 0 \leq 2^8 \frac{|y - y'|}{|y|^2} \,.$$

$\square$

11.2. Smooth functions.

LEMMA 11.2.1. *Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and differentiable, and let $n \in \mathbb{Z} \setminus \{0\}$. Then*

$$\hat{f}_n = \frac{1}{in} \widehat{f'}_n. \tag{11.2.1}$$

PROOF. This is part of the Lean library. $\square$

LEMMA 11.2.2. *Let $f : \mathbb{R} \to \mathbb{C}$ such that*

$$\sum_{n \in \mathbb{Z}} |\hat{f}_n| < \infty. \tag{11.2.2}$$

Then

$$\sup_{x \in [0, 2\pi]} |f(x) - S_N f(x)| \to 0 \tag{11.2.3}$$

as $N \to \infty$.

PROOF. This is part of the Lean library. $\square$

LEMMA 11.2.3. *Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and twice continuously differentiable. Then*

$$\sup_{x \in [0, 2\pi]} |f(x) - S_N f(x)| \to 0 \tag{11.2.4}$$

as $N \to \infty$.

PROOF. By Lemma 11.2.2, it suffices to show that the Fourier coefficients $\hat{f}_n$ are summable. Applying Lemma 11.2.1 twice and using the fact that f'' is continuous and thus bounded on $[0, 2\pi]$, we compute

$$\sum_{n \in \mathbb{Z}} |\hat{f}_n| = |\hat{f}_0| + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} |\widehat{f''}_n| \leq |\hat{f}_0| + \left(\sup_{x \in [0, 2\pi]} |f(x)| \right) \cdot \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} < \infty.$$

$\square$

PROOF. Lemma 11.1.2 now follows directly from the previous Lemma 11.2.3. □

11.3. The truncated Hilbert transform

Let M_n be the modulation operator acting on measurable 2π-periodic functions defined by

$$M_n g(x) = g(x)e^{inx}. \tag{11.3.1}$$

Define the approximate Hilbert transform by

$$L_N g = \frac{1}{N} \sum_{n=0}^{N-1} M_{-n-N} S_{N+n} M_{N+n} g. \tag{11.3.2}$$

LEMMA 11.3.1 (modulated averaged projection). *We have for every bounded measurable 2π-periodic function g*

$$\|L_N g\|_{L^2[-\pi,\pi]} \leq \|g\|_{L^2[-\pi,\pi]}. \tag{11.3.3}$$

PROOF. We have

$$\|M_n g\|_{L^2[-\pi,\pi]}^2 = \int_{-\pi}^{\pi} |e^{inx} g(x)|^2 \, dx = \int_{-\pi}^{\pi} |g(x)|^2 \, dx = \|g\|_{L^2[-\pi,\pi]}^2. \tag{11.3.4}$$

We have by the triangle inequality, the square root of the identity in (11.3.4), and Lemma 11.1.10

$$\|L_n g\|_{L^2[-\pi,\pi]} = \|\frac{1}{N} \sum_{n=0}^{N-1} M_{-n-N} S_{N+n} M_{N+n} g\|_{L^2[-\pi,\pi]}$$

$$\leq \frac{1}{N} \sum_{n=0}^{N-1} \|M_{-n-N} S_{N+n} M_{N+n} g\|_{L^2[-\pi,\pi]} = \frac{1}{N} \sum_{n=0}^{N-1} \|S_{N+n} M_{N+n} g\|_{L^2[-\pi,\pi]}$$

$$\leq \frac{1}{N} \sum_{n=0}^{N-1} \|M_{N+n} g\|_{L^2[-\pi,\pi]} = \frac{1}{N} \sum_{n=0}^{N-1} \|g\|_{L^2[-\pi,\pi]} = \|g\|_{L^2[-\pi,\pi]}. \tag{11.3.5}$$

This proves (11.3.4) and completes the proof of the lemma. □

LEMMA 11.3.2 (periodic domain shift). *Let f be a bounded 2π-periodic function. We have for any $0 \leq x \leq 2\pi$ that*

$$\int_0^{2\pi} f(y) \, dy = \int_{-x}^{2\pi-x} f(y) \, dy = \int_{-\pi}^{\pi} f(y-x) \, dy. \tag{11.3.6}$$

PROOF. We have by periodicity and change of variables

$$\int_{-x}^{0} f(y) \, dy = \int_{-x}^{0} f(y+2\pi) \, dy = \int_{2\pi-x}^{2\pi} f(y) \, dy. \tag{11.3.7}$$

We then have by breaking up the domain of integration and using (11.3.7)

$$\int_0^{2\pi} f(y) \, dy = \int_0^{2\pi-x} f(y) \, dy + \int_{2\pi-x}^{2\pi} f(y) \, dy$$

$$= \int_0^{2\pi-x} f(y) \, dy + \int_{-x}^{0} f(y) \, dy = \int_{-x}^{2\pi-x} f(y) \, dy. \tag{11.3.8}$$

This proves the first identity of the lemma. The second identity follows by substitution of y by $y - x$. □

LEMMA 11.3.3 (Young convolution). *Let f and g be two bounded non-negative measurable 2π-periodic functions on $\mathbb{R}$. Then*

$$\left(\int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(y)g(x-y)\,dy \right)^2 dx \right)^{\frac{1}{2}} \leq \|f\|_{L^2[-\pi,\pi]}\|g\|_{L^1[-\pi,\pi]}\,. \tag{11.3.9}$$

PROOF. Using Fubini and Lemma 11.3.2, we observe

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y)^2 g(x-y)\,dy\,dx = \int_{-\pi}^{\pi} f(y)^2 \int_{-\pi}^{\pi} g(x-y)\,dx\,dy$$

$$= \int_{-\pi}^{\pi} f(y)^2 \int_{-\pi}^{\pi} g(x)\,dx dy = \|f\|_{L^2[-\pi,\pi]}^2 \|g\|_{L^1[-\pi,\pi]}\,. \tag{11.3.10}$$

Let h be the nonnegative square root of g, then h is bounded and 2π-periodic with $h^2 = g$. We estimate the square of the left-hand side of (11.3.9) with Cauchy-Schwarz and then with (11.3.10) by

$$\int_{-\pi}^{\pi} (\int_{-\pi}^{\pi} f(y)h(x-y)h(x-y)\,dy)^2\,dx$$

$$\leq \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(y)^2 g(x-y)\,dy \right) \left(\int_{-\pi}^{\pi} g(x-y)\,dy \right)\,dx$$

$$= \|f\|_{L^2[-\pi,\pi]}^2 \|g\|_{L^1[-\pi,\pi]}^2\,.$$

Taking square roots, this proves the lemma. □

For $0 < r < 1$, Define the kernel k_r to be the 2π-periodic function

$$|k_r(x)| := \min\left(r^{-1}, 1 + \frac{r}{|1-e^{ix}|^2} \right)\,, \tag{11.3.11}$$

where the minimum is understood to be r^{-1} in case $1 = e^{ix}$.

LEMMA 11.3.4 (integrable bump convolution). *Let g, f be bounded measurable 2π-periodic functions. Let $0 < r < \pi$. Assume we have for all $0 \leq x \leq 2\pi$*

$$|g(x)| \leq k_r(x)\,. \tag{11.3.12}$$

Let

$$h(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,dy\,. \tag{11.3.13}$$

Then

$$\|h\|_{L^2[-\pi,\pi]} \leq 2^5 \|f\|_{L^2[-\pi,\pi]}\,. \tag{11.3.14}$$

PROOF. From monotonicity of the integral and (11.3.12),

$$\|g\|_{L^1[-\pi,\pi]} \leq \int_{-\pi}^{\pi} k_r(x)\,dx\,. \tag{11.3.15}$$

Using the symmetry $k_r(x) = k_r(-x)$, the assumption, and Lemma 11.1.9, the last display is equal to

$$= 2\int_0^\pi \min\left(\frac{1}{r}, 1 + \frac{r}{|1 - e^{ix}|^2}\right) dx$$

$$\leq 2\int_0^r \frac{1}{r}\, dx + 2\int_r^\pi 1 + \frac{64r}{x^2}\, dx$$

$$\leq 2 + 2\pi + 2\left(\frac{64r}{r} - \frac{64r}{\pi}\right) \leq 2^5. \tag{11.3.16}$$

Together with Lemma 11.3.3, this proves the lemma. □

LEMMA 11.3.5 (Dirichlet approximation). *Let $0 < r < 1$. Let N be the smallest integer larger than $\frac{1}{r}$. There is a 2π-periodic continuous function L' on $\mathbb{R}$ that satisfies for all $-\pi \leq x \leq \pi$ and all 2π-periodic bounded measurable functions f on $\mathbb{R}$*

$$L_N f(x) = \frac{1}{2\pi}\int_{-\pi}^\pi f(y) L'(x - y)\, dy \tag{11.3.17}$$

and

$$\left|L'(x) - \mathbf{1}_{\{y: r < |y| < 1\}}\kappa(x)\right| \leq 2^5 k_r(x). \tag{11.3.18}$$

PROOF. We have by definition and Lemma 11.1.8

$$L_N g(x) = \frac{1}{N}\sum_{n=0}^{N-1}\int_{-\pi}^\pi e^{-i(N+n)x} K_{N+n}(x - y) e^{i(N+n)y} g(y)\, dy. \tag{11.3.19}$$

This is of the form (11.3.17) with the continuous function

$$L'(x) = \frac{1}{N}\sum_{n=0}^{N-1} K_{N+n}(x) e^{-i(N+n)x}. \tag{11.3.20}$$

With (11.1.22) of Lemma 11.1.8 we have $|K_N(x)| \leq N$ for every x and thus

$$|L'(x)| \leq \frac{1}{N}\sum_{n=0}^{N-1}(N + n) \leq 2N \leq 2^2 r^{-1}. \tag{11.3.21}$$

Therefore, for $|x| \in [0, r) \cup (1, \pi]$, we have

$$\left|L'(x) - \mathbf{1}_{\{y: r < |y| < 1\}}(x)\kappa(x)\right| = |L'(x)| \leq 2^2 r^{-1}. \tag{11.3.22}$$

This proves (11.3.18) for $|x| \in [0, r)$ since $k_r(x) = r^{-1}$ in this case.

For $e^{ix'} \neq 1$ and may use the expression (11.1.23) for K_N in Lemma 11.1.8 to obtain

$$L'(x) = \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{e^{i(N+n)x}}{1 - e^{-ix}} + \frac{e^{-i(N+n)x}}{1 - e^{ix}}\right) e^{-i(N+n)x}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{1}{1 - e^{-ix}} + \frac{e^{-i2(N+n)x}}{1 - e^{ix}}\right)$$

$$= \frac{1}{1 - e^{-ix}} + \frac{1}{N}\frac{e^{-i2Nx}}{1 - e^{ix}}\sum_{n=0}^{N-1} e^{-i2nx} \tag{11.3.23}$$

and thus

$$L'(x) - \mathbf{1}_{\{y:\, r<|y|<1\}}\kappa(x) = L''(x) + \frac{1 - \mathbf{1}_{\{y:\, r<|y|<1\}}(x)(1-|x|)}{1-e^{-ix}}, \qquad (11.3.24)$$

where

$$L''(x) := \frac{1}{N}\frac{e^{-i2Nx}}{1-e^{ix}}\sum_{n=0}^{N-1}e^{-i2nx}.$$

For $x \in [-\pi, r] \cup [r, \pi]$, we have using Lemma 11.1.9 that

$$\left|\frac{1 - \mathbf{1}_{\{y:\, r<|y|<1\}}(x)(1-|x|)}{1-e^{-ix}}\right| = \left|\frac{\min(|x|,1)}{1-e^{-ix}}\right| \leq \frac{8\min(|x|,1)}{|x|}$$

$$\leq 2^3 \cdot 1 \leq 2^3 k_r(x). \qquad (11.3.25)$$

Next, we need to estimate $L''(x)$. If the real part of e^{ix} is negative, we have

$$1 \leq |1-e^{ix}| \leq 2. \qquad (11.3.26)$$

and hence

$$|L''(x)| \leq \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1 \leq 1 + \frac{r}{|1-e^{ix}|^2}. \qquad (11.3.27)$$

If the real part of e^{ix} is positive and in particular while still $e^{ix} \neq \pm 1$, then we have by telescoping

$$(1-e^{-2ix})\sum_{n=0}^{N-1}e^{-i2nx} = 1 - e^{-i2Nx}. \qquad (11.3.28)$$

As $e^{-2ix} \neq 1$, we may divide by $1-e^{-2ix}$ and insert this into (11.3.23) to obtain

$$L''(x) = \frac{1}{N}\frac{e^{-i2Nx}}{1-e^{ix}}\frac{1-e^{-i2Nx}}{1-e^{-2ix}}. \qquad (11.3.29)$$

Hence, with Lemma 11.1.9 and nonnegativity of the real part of e^{ix}

$$|L''(x)| \leq \frac{2}{N}\frac{1}{|1-e^{ix}|}\frac{1}{|1-e^{-2ix}|}$$

$$= \frac{2}{N}\frac{1}{|1-e^{ix}|^2}\frac{1}{|1+e^{ix}|} \leq \frac{4r}{|1-e^{ix}|^2} \leq 2^2\left(1 + \frac{r}{|1-e^{ix}|^2}\right) \qquad (11.3.30)$$

Inequalities (11.3.21), (11.3.22), (11.3.24), (11.3.25), (11.3.27), and (11.3.30) prove (11.3.18). This completes the proof of the lemma. □

We now prove Lemma 11.1.5.

PROOF OF LEMMA 11.1.5. We first show that if f is supported in $[-3/2, 3/2]$, then

$$\|H_r f\|_{L^2(\mathbb{R})} \leq 2^{16}\|f\|_{L^2(\mathbb{R})}. \qquad (11.3.31)$$

Let $\tilde{f}$ be the 2π-periodic extension of f to $\mathbb{R}$. Let N be the smallest integer larger than $\frac{1}{r}$. Then, by Lemma 11.3.5 and the triangle inequality, for $x \in [-\pi, \pi]$ we have

$$|H_r\tilde{f}(x)| \leq 2\pi|L_N\tilde{f}(x)| + 2^5\left|\int_{-\pi}^{\pi}k_r(x-y)\tilde{f}(y)\,dy\right|.$$

Taking L^2 norm over the interval $[-\pi, \pi]$ and using its sub-additivity, we get

$$\|H_r \tilde{f}\|_{L^2([-\pi,\pi])}$$

$$\leq 2\pi \|L_N \tilde{f}\|_{L^2([-\pi,\pi])} + 2^5 \left(\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} k_r(x-y)\tilde{f}(y)\,dy \right|^2 dx \right)^{\frac{1}{2}}.$$

Since κ is supported in $[-1, 1]$, we have that $H_r f$ is supported in $[-5/2, 5/2]$ and agrees there with $H_r \tilde{f}(x)$. Using Lemma 11.3.1 and Lemma 11.3.4, we conclude

$$\|H_r f\|_{L^2(\mathbb{R})} \leq \|H_r \tilde{f}\|_{L^2([-\pi,\pi])} \leq 2\pi \|f\|_{L^2(\mathbb{R})} + 2^{10}\|f\|_{L^2(\mathbb{R})}, \tag{11.3.32}$$

which gives (11.3.31).

Suppose now that f is supported in $[c, c+3]$ for some $c \in \mathbb{R}$. Then the function $g(x) = f(c + \frac{3}{2} + x)$ is supported in $[-3/2, 3/2]$. By a change of variables in (11.1.17), we have $H_r g(x) = H_r f(c + 3/2 + x)$. Thus, by (11.3.31)

$$\|H_r g\|_2 = \|H_r f\|_2 \leq 2^{11}\|f\|_2 = \|g\|_2. \tag{11.3.33}$$

Let now f be arbitrary. Since $\kappa(x) = 0$ for $|x| > 1$, we have for all $x \in [c+1, c+2]$

$$H_r f(x) = H_r(f \mathbf{1}_{[c,c+3]})(x).$$

Thus

$$\int_{c+1}^{c+2} |H_r f(x)|^2 \,dx \leq \int_{\mathbb{R}} |H_r(f \mathbf{1}_{[c,c+3]})(x)|^2 \,dx.$$

Applying the bound (11.3.33), this is

$$\leq 2^{11} \int_c^{c+3} |f(x)|^2 \,dx.$$

Summing over all $c \in \mathbb{Z}$, we obtain

$$\int_{\mathbb{R}} |H_r f(x)|^2 \,dx \leq 3 \cdot 2^{11} \int_{\mathbb{R}} |f(x)|^2 \,dx.$$

This completes the proof. $\qquad\square$

11.4. The proof of the van der Corput Lemma

PROOF OF LEMMA 11.1.6. Let g be a Lipschitz continuous function as in the lemma. Assume first that $n = 0$. Then

$$\int_{\alpha}^{\beta} g(x)\,dx \leq |\beta - \alpha| \sup_{\alpha \leq x \leq \beta} |g(x)| \leq |\beta - \alpha| \|g\|_{Lip(\alpha,\beta)} (1 + |n||\beta - \alpha|)^{-1}$$

Assume now $n \neq 0$. Without loss of generality, we may assume $n > 0$. We distinguish two cases. If $\beta - \alpha < \frac{\pi}{n}$, we have by the triangle inequality

$$\left| \int_{\alpha}^{\beta} g(x)e^{inx}\,dx \right| \leq |\beta - \alpha| \sup_{x \in [\alpha,\beta]} |g(x)| \leq 2\pi |\beta - \alpha| \|g\|_{Lip(\alpha,\beta)} (1 + |n||\beta - \alpha|)^{-1}.$$

We turn to the case $\frac{\pi}{n} \leq \beta - \alpha$. We have

$$e^{in(x+\pi/n)} = -e^{inx}.$$

Using this, we write

$$\int_\alpha^\beta g(x)e^{inx}\,\mathrm{d}x = \frac{1}{2}\int_\alpha^\beta g(x)e^{inx}\,\mathrm{d}x - \frac{1}{2}\int_\alpha^\beta g(x)e^{in(x+\pi/n)}\,\mathrm{d}x\,.$$

We split the the first integral at $\alpha + \frac{\pi}{n}$ and the second one at $\beta - \frac{\pi}{n}$, and make a change of variables in the second part of the first integral to obtain

$$= \frac{1}{2}\int_\alpha^{\alpha+\frac{\pi}{n}} g(x)e^{inx}\,\mathrm{d}x - \frac{1}{2}\int_{\beta-\frac{\pi}{n}}^\beta g(x)e^{in(x+\pi/n)}\,\mathrm{d}x$$

$$+ \frac{1}{2}\int_{\alpha+\frac{\pi}{n}}^\beta (g(x) - g(x - \frac{\pi}{n}))e^{inx}\,\mathrm{d}x\,.$$

The sum of the first two terms is by the triangle inequality bounded by

$$\frac{\pi}{n}\sup_{x\in[\alpha,\beta]} |g(x)|\,.$$

The third term is by the triangle inequality at most

$$\frac{1}{2}\int_{\alpha+\frac{\pi}{n}}^\beta |g(x) - g(x - \frac{\pi}{n})|\,\mathrm{d}x$$

$$\leq \frac{|\beta - \alpha|}{2}\frac{\pi}{n}\sup_{\alpha\leq x<y\leq\beta} \frac{|g(x) - g(y)|}{|x - y|}\,.$$

Adding the two terms, we obtain

$$\left|\int_\alpha^\beta g(x)e^{-inx}\,\mathrm{d}x\right| \leq \frac{\pi}{n}\|g\|_{\mathrm{Lip}(\alpha,\beta)}\,.$$

This completes the proof of the lemma, using that with $\frac{\pi}{n}\leq\beta-\alpha$,

$$\frac{\pi}{n} = \frac{2\pi|\beta-\alpha|}{2n|\beta-\alpha|} \leq 2\pi|\beta-\alpha|(1+n|\beta-\alpha|)^{-1}\,.$$

$\square$

11.5. Partial sums as orthogonal projections

This subsection proves Lemma 11.1.10

LEMMA 11.5.1 (partial sum projection). *Let f be a bounded 2π-periodic measurable function. Then, for all $N\geq 0$*

$$S_N(S_N f) = S_N f\,. \tag{11.5.1}$$

PROOF. Let $N > 0$ be given. With K_N as in Lemma 11.1.8,

$$S_N(S_N f)(x) = \frac{1}{2\pi}\int_0^{2\pi} S_N f(y)K_N(x - y)\,dy$$

$$= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(y')K_N(y - y')K_N(x - y)\,dy'dy\,. \tag{11.5.2}$$

We have by Lemma 11.1.8

$$\frac{1}{2\pi} \int_0^{2\pi} K_N(y - y')K_N(x - y)\, dy$$

$$= \frac{1}{2\pi} \sum_{n=-N}^{N} \sum_{n'=-N}^{N} \int_0^{2\pi} e^{in(y-y')}e^{in'(x-y)}\, dy$$

$$= \frac{1}{2\pi} \sum_{n=-N}^{N} \sum_{n'=-N}^{N} e^{i(n'x-ny')} \int_0^{2\pi} e^{i(n-n')y}\, dy\,. \tag{11.5.3}$$

By Lemma 11.1.7, the summands for $n \neq n'$ vanish. We obtain for (11.5.3)

$$= \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-y')} \int_0^{2\pi} dy = K_N(x - y')\,. \tag{11.5.4}$$

Applying Fubini in (11.5.2) and using (11.5.4) gives

$$S_N(S_N f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y')K(x - y')\, dy' = S_N f(x) \tag{11.5.5}$$

This proves the lemma. $\qquad\square$

LEMMA 11.5.2 (partial sum selfadjoint). *We have for any 2π-periodic bounded measurable g, f that*

$$\int_0^{2\pi} \overline{S_N f(x)}g(x) = \int_0^{2\pi} \overline{f(x)}S_N g(x)\, dx\,. \tag{11.5.6}$$

PROOF. We have with K_N as in Lemma 11.1.8 for every x

$$\overline{K_N(x)} = \sum_{n=-N}^{N} \overline{e^{inx}} = \sum_{n=-N}^{N} e^{-inx} = K_N(-x)\,. \tag{11.5.7}$$

Further, with Lemma 11.1.8 and Fubini

$$\int_0^{2\pi} \overline{S_N f(x)}g(x) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\pi}^{\pi} \overline{f(y)K_N(x - y)}g(x)\, dydx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\pi}^{\pi} \overline{f(y)}K_N(y - x)g(x)\, dxdy = \int_0^{2\pi} \overline{f(x)}S_N g(x)\, dx\,. \tag{11.5.8}$$

This proves the lemma. $\qquad\square$

We turn to the proof of Lemma 11.1.10.

We have with Lemma 11.5.2, then Lemma 11.5.1 and the Lemma 11.5.2 again

$$\int_0^{2\pi} S_N f(x)\overline{S_N f(x)}\, dx \int_0^{2\pi} f(x)\overline{S_N(S_N f)(x)}\, dx$$

$$= \int_0^{2\pi} f(x)\overline{S_N f(x)}\, dx = \int_0^{2\pi} S_N f(x)\overline{f(x)}\, dx\,. \tag{11.5.9}$$

We have by the distributive law

$$\int_0^{2\pi} (f(x) - S_N f(x))(\overline{f(x) - S_N f(x)}) \, dx = \tag{11.5.10}$$

$$\int_0^{2\pi} f(x)\overline{f(x)} - S_N f(x)\overline{f(x)} - f(x)\overline{S_N f(x)} + S_N f(x)\overline{S_N f(x)} \, dx$$

Using the various identities expressed in (11.5.9), this becomes

$$= \int_0^{2\pi} f(x)\overline{f(x)} \, dx - \int_0^{2\pi} S_N f(x)\overline{S_N f(x)} \, dx. \tag{11.5.11}$$

As (11.5.10) has nonnegative integrand and is thus nonnegative, we conclude

$$\int_0^{2\pi} S_N f(x)\overline{S_N f(x)} \, dx \leq \int_0^{2\pi} f(x)\overline{f(x)} \, dx. \tag{11.5.12}$$

As both sides are positive, we may take the square root of this inequality. This completes the proof of the lemma.

11.6. The error bound

LEMMA 11.6.1 (Dirichlet kernel - Hilbert kernel relation). *For all $N \in \mathbb{Z}$ and $x \in [-\pi, \pi] \smallsetminus \{0\}$,*

$$\left| K_N(x) - (e^{-iNx}\kappa(x) + \overline{e^{-iNx}\kappa(x)}) \right| \leq \pi.$$

PROOF. Let $N \in \mathbb{Z}$ and $x \in [-\pi, \pi] \smallsetminus \{0\}$. With Lemma 11.1.8, we obtain

$$K_N(x) - (e^{-iNx}\kappa(x) + \overline{e^{-iNx}\kappa(x)}) = e^{-iNx}\frac{\min(|x|, 1)}{1 - e^{ix}} + e^{iNx}\frac{\min(|x|, 1)}{1 - e^{-ix}}.$$

Using Lemma 11.1.9 with $\eta = \min(|x|, 1)$, we bound

$$\left| K_N(x) - (e^{-iNx}\kappa(x) + \overline{e^{-iNx}\kappa(x)}) \right| \leq \frac{\min(|x|, 1)}{|1 - e^{ix}|} + \frac{\min(|x|, 1)}{|1 - e^{-ix}|} \leq \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

$\square$

LEMMA 11.6.2 (partial Fourier sum bound). *Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable 2π-periodic function such that for some $\delta > 0$ and every $x \in \mathbb{R}$,*

$$|g(x)| \leq \delta. \tag{11.6.1}$$

Then for every $x \in [0, 2\pi]$ and $N > 0$,

$$|S_N g(x)| \leq \frac{1}{2\pi}(Tg(x) + T\bar{g}(x)) + \pi\delta.$$

PROOF. Let $x \in [0, 2\pi]$ and $N > 0$. We have with Lemma 11.1.8

$$|S_N g(x)| = \frac{1}{2\pi}\left| \int_0^{2\pi} g(y)K_N(x - y) \, dy \right|.$$

We use 2π-periodicity of g and K_N to shift the domain of integration to obtain

$$= \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) K_N(x-y) \, dy \right| .$$

Using the triangle inequality, we split this as

$$\leq \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) \left(K_N(x-y) - \max(|x-y|, 0) K_N(x-y) \right) \, dy \right| \qquad (11.6.2)$$

$$+ \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) \max(|x-y|, 0) K_N(x-y) \, dy \right| . \qquad (11.6.3)$$

Note that all integrals are well defined, since K_N is by (11.1.22) bounded by $2N+1$. Using that

$$\max(|x-y|, 0) K_N(x-y) = e^{-iN(x-y)} \kappa(x-y) + \overline{e^{-iN(x-y)} \kappa(x-y)}, \qquad (11.6.4)$$

Lemma 11.6.1 and (11.6.5), we bound (11.6.2) by

$$\frac{1}{2\pi} \int_{x-\pi}^{x+\pi} |g(y)| \left| K_N(x-y) - e^{-iN(x-y)} \kappa(x-y) + \overline{e^{-iN(x-y)} \kappa(x-y)} \right| dy \leq \pi\delta .$$

By dominated convergence and since $\kappa(x-y) = 0$ for $|x-y| > 1$, (11.6.3) equals

$$\frac{1}{2\pi} \lim_{r \to 0^+} \left| \int_{r < |x-y| < 1} g(y) \max(|x-y|, 0) K_N(x-y) \, dy \right| .$$

We bound the limit by a supremum and rewrite using (11.6.4),

$$\leq \frac{1}{2\pi} \sup_{r>0} \left| \int_{r < |x-y| < 1} g(y) \left(e^{-iN(x-y)} \kappa(x-y) + \overline{e^{-iN(x-y)} \kappa(x-y)} \right) dy \right|$$

Using the triangle inequality, we further bound this by

$$\leq \frac{1}{2\pi} \sup_{r>0} \left| \int_{r < |x-y| < 1} g(y) e^{-iNy} \kappa(x-y) \, dy \right|$$

$$+ \frac{1}{2\pi} \sup_{r>0} \left| \int_{r < |x-y| < 1} \overline{g}(y) e^{-iNy} \kappa(x-y) \, dy \right| .$$

By the definition (11.1.15) of T, this is

$$\leq \frac{1}{2\pi} \left(Tg(x) + T\overline{g}(x) \right) .$$

$\square$

LEMMA 11.6.3 (real Carleson operator measurable). *Let f be a bounded measurable function on $\mathbb{R}$. Then Tf as defined in (11.1.15) is measurable.*

PROOF. Since a countable supremum of measurable functions is measurable, it suffices to show that for every $n \in \mathbb{Z}$,

$$x \mapsto \sup_{r>0} \left| \int_{r < |x-y| < 1} f(y) \kappa(x-y) e^{iny} \, dy \right|$$

is measurable. So let $n \in \mathbb{Z}$. Note that for each $x \in \mathbb{R}$, the function

$$r \mapsto \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right|$$

is continuous on $(0,\infty)$ since the integrand is locally bounded on the domain $0 < |x-y| < 1$ by the assumptions on f and Lemma 11.1.11. Thus, for each $x \in \mathbb{R}$,

$$\sup_{r>0} \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right| = \sup_{r \in \mathbb{Q}_{>0}} \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right|$$

The right hand side is again a countable supremum so it remains to show that for every $r \in \mathbb{Q}_{>0}$,

$$x \mapsto \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right| = \left| \int \mathbf{1}_{\{r<|x-\cdot|<1\}}(y)f(y)\kappa(x-y)e^{iny}\, dy \right|$$

is measurable, which follows from the fact that the integrand is measurable in (x,y).
□

LEMMA 11.6.4 (partial Fourier sums of small). *Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable 2π-periodic function such that for some $\delta > 0$ and every $x \in \mathbb{R}$,*

$$|g(x)| \leq \delta. \tag{11.6.5}$$

Then for every $\epsilon > 0$, there exists a measurable set $E \subset [0,2\pi]$ with $|E| < \epsilon$ such that for every $x \in [0,2\pi] \smallsetminus E$ and $N > 0$,

$$|S_N g(x)| \leq C_\epsilon \delta, \tag{11.6.6}$$

where

$$C_\epsilon = \left(\frac{8}{\pi\epsilon} \right)^{\frac{1}{2}} C_{4,2} + \pi. \tag{11.6.7}$$

PROOF. Define

$$E := \{ x \in [0,2\pi] \ : \ \sup_{N>0} |S_N g(x)| > C_\epsilon \delta \}.$$

Then (11.6.6) clearly holds, and it remains to show that $|E| \leq \epsilon$. Using Lemma 11.6.2, we obtain

$$E \subset \{ x \in [0,2\pi] \ : \ C_\epsilon \delta < \frac{1}{2\pi}(Tg(x) + T\bar{g}(x)) + \pi\delta \} \subset E_1 \cup E_2,$$

where

$$E_1 := \{ x \in [0,2\pi] \ : \ \pi(C_\epsilon - \pi)\delta < Tg(x) \}$$
$$E_2 := \{ x \in [0,2\pi] \ : \ \pi(C_\epsilon - \pi)\delta < T\bar{g}(x) \}.$$

By Lemma 11.6.3, E_1 and E_2 are measurable. Thus,

$$\pi(C_\epsilon - \pi)\delta|E_1| \leq \int_{E_1} Tg(x)\, dx = \delta \int_{E_1} T(\delta^{-1}g\mathbf{1}_{[-\pi,3\pi]})(x)\, dx.$$

Applying Lemma 11.1.4 with $F = [-\pi,3\pi]$ and $G = E'$, it follows that this is

$$\leq \delta \cdot C_{4,2}|F|^{\frac{1}{2}}|E_1|^{\frac{1}{2}} \leq (4\pi)^{\frac{1}{2}} C_{4,2}\delta \cdot |E'|^{\frac{1}{2}}.$$

Rearranging, we obtain

$$|E_1| \leq \left(\frac{(4\pi)^{\frac{1}{2}} C_{4,2}}{\pi(C_\epsilon - \pi)} \right)^2 = \frac{\epsilon}{2}.$$

Analogously, we get the same estimate for $|E_2|$. This completes the proof using $|E| \leq |E_1| + |E_2|$. $\qquad\square$

PROOF OF LEMMA 11.1.3. Lemma 11.1.3 now follows directly from Lemma 11.6.4 with $\delta := \epsilon'$. $\qquad\square$

11.7. Carleson on the real line

We prove Lemma 11.1.4.

Consider the standard distance function

$$\rho(x, y) = |x - y| \qquad\qquad (11.7.1)$$

on the real line $\mathbb{R}$.

LEMMA 11.7.1 (real line metric). *The space $(\mathbb{R}, \rho)$ is a complete locally compact metric space.*

PROOF. This is part of the Lean library. $\qquad\square$

LEMMA 11.7.2 (real line ball). *For $x \in R$ and $R > 0$, the ball $B(x, R)$ is the interval $(x - R, x + R)$*

PROOF. Let $x' \in B(x, R)$. By definition of the ball, $|x' - x| < R$. It follows that $x' - x < R$ and $x - x' < R$. It follows $x' < x + R$ and $x' > x - R$. This implies $x' \in (x - R, x + R)$. Conversely, let $x' \in (x - R, x + R)$. Then $x' < x + R$ and $x' > x - R$. It follows that $x' - x < R$ and $x - x' < R$. It follows that $|x' - x| < R$, hence $x' \in B(x, R)$. This proves the lemma. $\qquad\square$

We consider the Lebesgue measure μ on $\mathbb{R}$.

LEMMA 11.7.3 (real line measure). *The measure μ is a sigma-finite non-zero Radon-Borel measure on $\mathbb{R}$.*

PROOF. This is part of the Lean library. $\qquad\square$

LEMMA 11.7.4 (real line ball measure). *We have for every $x \in \mathbb{R}$ and $R > 0$*

$$\mu(B(x, R)) = 2R. \qquad\qquad (11.7.2)$$

PROOF. We have with Lemma 11.7.2

$$\mu(B(x, R)) = \mu((x - R, x + R)) = 2R. \qquad\qquad (11.7.3)$$

$\qquad\square$

LEMMA 11.7.5 (real line doubling). *We have for every $x \in \mathbb{R}$ and $R > 0$*

$$\mu(B(x, 2R)) = 2\mu(B(x, R)). \qquad\qquad (11.7.4)$$

Proof. We have with Lemma 11.7.4

$$\mu(B(x, 2R) = 4R = 2\mu(B(x, R)).$$
(11.7.5)

This proves the lemma.   □

The preceding four lemmas show that $(\mathbb{R}, \rho, \mu, 4)$ is a doubling metric measure space. Indeed, we even show that $(\mathbb{R}, \rho, \mu, 1)$ is a doubling metric measure space, but we may relax the estimate in Lemma 11.7.5 to conclude that $(\mathbb{R}, \rho, \mu, 4)$ is a doubling metric measure space.

For each $n \in \mathbb{Z}$ define $\vartheta_n : \mathbb{R} \to \mathbb{R}$ by

$$\vartheta_n(x) = nx.$$
(11.7.6)

Let Θ be the collection $\{\vartheta_n, n \in \mathbb{Z}\}$. Note that for every $n \in \mathbb{Z}$ we have $\vartheta_n(0) = 0$. Define

$$d_{B(x,R)}(\vartheta_n, \vartheta_m) := 2R|n - m|.$$
(11.7.7)

Lemma 11.7.6 (frequency metric). *For every $R > 0$ and $x \in X$, the function $d_{B(x,R)}$ is a metric on Θ.*

Proof. This follows immediately from the fact that the standard metric on $\mathbb{Z}$ is a metric.   □

Lemma 11.7.7 (oscillation control). *For every $R > 0$ and $x \in X$, and for all $n, m \in \mathbb{Z}$, we have*

$$\sup_{y,y' \in B(x,R)} |ny - ny' - my + my'| \leq 2|n - m|R.$$
(11.7.8)

Proof. The right hand side of (11.7.8) equals

$$\sup_{y,y' \in B(x,R)} |(n - m)(y - x) - (n - m)(y' - x)|.$$

The lemma then follows from the triangle inequality.   □

Lemma 11.7.8 (frequency monotone). *For any $x, x' \in X$ and $R, R' > 0$ with $B(x, R) \subset B(x, R')$, and for any $n, m \in \mathbb{Z}$*

$$d_{B(x,R)}(\vartheta_n, \vartheta_m) \leq d_{B(x',R')}(\vartheta_n, \vartheta_m).$$

Proof. This follows immediately from the definition (11.7.7) and $R \leq R'$.   □

Lemma 11.7.9 (frequency ball doubling). *For any $x, x' \in \mathbb{R}$ and $R > 0$ with $x \in B(x', 2R)$ and any $n, m \in \mathbb{Z}$, we have*

$$d_{B(x',2R)}(\vartheta_n, \vartheta_m) \leq 2d_{B(x,R)}(\vartheta_n, \vartheta_m).$$
(11.7.9)

Proof. With (11.7.7), both sides of (11.7.9) are equal to $4R|n - m|$. This proves the lemma.   □

Lemma 11.7.10 (frequency ball growth). *For any $x, x' \in \mathbb{R}$ and $R > 0$ with $B(x, R) \subset B(x', 2R)$ and any $n, m \in \mathbb{Z}$, we have*

$$2d_{B(x,R)}(\vartheta_n, \vartheta_m) \leq d_{B(x',2R)}(\vartheta_n, \vartheta_m).$$
(11.7.10)

PROOF. With (11.7.7), both sides of (11.7.9) are equal to $4R|n-m|$. This proves the lemma. $\qquad\square$

LEMMA 11.7.11 (integer ball cover). *For every $x \in \mathbb{R}$ and $R > 0$ and every $n \in \mathbb{Z}$ and $R' > 0$, there exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that*

$$B' \subset B_1 \cup B_2 \cup B_3\,, \tag{11.7.11}$$

where

$$B' = \{\vartheta \in \Theta : d_{B(x,R)}(\vartheta, \vartheta_n) < 2R'\} \tag{11.7.12}$$

and for $j = 1, 2, 3$

$$B_j = \{\vartheta \in \Theta : d_{B(x,R)}(\vartheta, \vartheta_{m_j}) < R'\}\,. \tag{11.7.13}$$

PROOF. Let m_1 be the largest integer smaller than or equal to $n - \frac{R'}{2R}$. Let $m_2 = n$. Let m_3 be the smallest integer larger than or equal to $n + \frac{R'}{2R}$.

Let $\vartheta_{n'} \in B'$, then with (11.7.7), we have

$$2R|n - n'| < 2R'\,. \tag{11.7.14}$$

Assume first $n' \le m_1$. With (11.7.14) we have

$$R|m_1 - n'| = R(m_1 - n') = R(m_1 - n) + R(n - n')$$

$$< -\frac{R'}{2} + R' = -\frac{R'}{2}\,. \tag{11.7.15}$$

We conclude $\vartheta_{n'} \in B_1$.

Assume next $m_1 < n' < m_3$. Then $\vartheta_{n'} \in B_2$.

Assume finally that $m_3 \le n'$. With (11.7.14) we have

$$R|m_3 - n'| = R(n' - m_3) = R(n' - n) + R(n - m_3)$$

$$< R' - \frac{R'}{2} = -\frac{R'}{2}\,. \tag{11.7.16}$$

We conclude $\vartheta_{n'} \in B_1$. This completes the proof of the lemma. $\qquad\square$

LEMMA 11.7.12 (real van der Corput). *For any $x \in \mathbb{R}$ and $R > 0$ and any function $\varphi : X \to \mathbb{C}$ supported on $B' = B(x, R)$ such that*

$$\|\varphi\|_{\mathrm{Lip}(B')} = \sup_{x \in B'} |\varphi(x)| + R \sup_{x,y \in B', x \ne y} \frac{|\varphi(x) - \varphi(y)|}{\rho(x, y)} \tag{11.7.17}$$

is finite and for any $n, m \in \mathbb{Z}$, we have

$$\left| \int_{B'} e(\vartheta_n(x) - \vartheta_m(x))\varphi(x)d\mu(x) \right| \le 2\pi\mu(B') \frac{\|\varphi\|_{\mathrm{Lip}(B')}}{1 + d_{B'}(\vartheta_n, \vartheta_m)}\,. \tag{11.7.18}$$

PROOF. Set $n' = n - m$. Then we have to prove

$$\left| \int_{x-R}^{x+R} e^{in'y}\varphi(y)dy \right| \le 4\pi R\|\varphi\|_{\mathrm{Lip}(B')}(1 + 2R|n'|)^{-1}\,. \tag{11.7.19}$$

This follows from Lemma 11.1.6 with $\alpha = x - R$ and $\beta = x + R$. $\qquad\square$

Proof of Lemma 11.1.4. The preceding chain of lemmas establishes that Θ is a cancellative, compatible collection of functions on $(\mathbb{R}, \rho, \mu, 4)$. Again, some of the statements in these lemmas are stronger than what is needed for $a = 4$, but can be relaxed to give the desired conclusion for $a = 4$.

With κ as near (11.1.12), define the function $K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ as in Theorem 1.0.2 by

$$K(x, y) := \kappa(x - y) . \tag{11.7.20}$$

The function K is continuous outside the diagonal $x = y$ and vanishes on the diagonal. Hence it is measurable.

By Lemmas 11.1.11 and 11.1.12, it follows that K is a two-sided Calderón–Zygmund kernel on $(\mathbb{R}, \rho, \mu, 4)$. Lemma 11.1.5 verifies (10.0.3). Thus the assumptions of Theorem 10.0.1 are all satisfied. Applying the Theorem, Lemma 11.1.4 follows.

$\square$

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