Higher Groups in Homotopy Type Theory

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Homotopy Hypothesis

The homotopy hypothesis states:

$$\text{homotopy } n\text{-types} \qquad \simeq \qquad n\text{-groupoids}$$

$$(n \in \mathbb{N} \text{ or } n = \infty)$$

Depending on the setting, this can be a theorem, conjecture or axiom.

In homotopy type theory, the types correspond to homotopy types, so we can study the homotopy hypothesis in HoTT.

Truncated and Connected Types

In HoTT types can be truncated (have trivial high-dimensional structure):

$$\begin{aligned} \mathrm{istrunc}_{-2}\, A &:= \mathrm{iscontr}\, A := (a : A) \times \big((x : A) \to (a = x)\big) \\ \mathrm{istrunc}_{n+1}\, A &:= (x\ y : A) \to \mathrm{istrunc}_n(x = y) \end{aligned}$$

The truncation $\|A\|_n$ is the universal n-truncated approximation of A.

We then also get connected types (have trivial low-dimensional structure):

$$\mathrm{isconn}_n\, A := \mathrm{iscontr}\, \|A\|_n$$

We define universes of pointed/truncated/connected types:

$$\begin{aligned} \mathrm{Type}_{\mathrm{pt}} &:= (A : \mathrm{Type}) \times (\mathrm{pt} : A) \\ \mathrm{Type}^{\le n} &:= (A : \mathrm{Type}) \times \mathrm{istrunc}_n\, A \\ \mathrm{Type}^{> n} &:= (A : \mathrm{Type}) \times \mathrm{isconn}_n\, A \end{aligned}$$

Loop Spaces

A pointed (0-)connected type $B : \mathrm{Type}^{>0}_{\mathrm{pt}}$ can be viewed as presenting a higher group, with carrier

$$\Omega B :\equiv (\mathrm{pt} =_B \mathrm{pt}).$$

The group structure on ΩB is induced form the identity type:

- Multiplication is path concatenation
- Inversion is path inversion
- The unit is the constant path
- Higher group laws correspond to higher coherences for paths.

Higher Groups

Switching perspective, we can define a higher group to be a carrier $G : \mathrm{Type}$ with a choice of delooping $BG : \mathrm{Type}$.

$$\begin{aligned}\infty\text{-Group} &:= (G : \mathrm{Type}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G \simeq \Omega BG) \\ &\simeq (G : \mathrm{Type}_{\mathrm{pt}}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G \simeq_{\mathrm{pt}} \Omega BG) \\ &\simeq \mathrm{Type}_{\mathrm{pt}}^{>0}\end{aligned}$$

We can define n-groups by assuming that the carrier is truncated G.

$$\begin{aligned}n\text{-Group} &:= (G : \mathrm{Type}_{\mathrm{pt}}^{<n}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G \simeq_{\mathrm{pt}} \Omega BG) \\ &\simeq \mathrm{Type}_{\mathrm{pt}}^{>0,\leq n}\end{aligned}$$

k-tuply Groupal Groupoids

Higher loop spaces are better-behaved. For example:

Theorem (Eckmann-Hilton)

For $p, q : \Omega^2 A$ we have

$$p \cdot q = q \cdot p.$$

If the carrier G of an $(n+1)$-group has k-fold deloopings, we say it is a k-tuply groupal n-groupoid.

$$\begin{aligned}
(n,k)\mathrm{GType} &:= (G : \mathrm{Type}_{\mathrm{pt}}^{\le n}) \times (B^k G : \mathrm{Type}_{\mathrm{pt}}^{\ge k}) \times (G \simeq_{\mathrm{pt}} \Omega^k B^k G) \\
&\simeq \mathrm{Type}_{\mathrm{pt}}^{\ge k, \le n+k} \\
(n,\omega)\mathrm{GType} &:= \lim_k (n,k)\mathrm{GType} \\
&\simeq \big(B^- G : (k : \mathbb{N}) \to \mathrm{Type}_{\mathrm{pt}}^{\ge k, \le n+k}\big) \\
&\qquad \times \big((k : \mathbb{N}) \to B^k G \simeq_{\mathrm{pt}} \Omega B^{k+1} G\big).
\end{aligned}$$

Periodic Table of Higher Groups

Table: Periodic table of k-tuply groupal n-groupoids, $(n,k)\mathrm{GType}$.

$k \setminus n$	0	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	3-group	$\cdots$	∞-group
2	abelian group	braided 2-group	braided 3-group	$\cdots$	braided ∞-group
3	— " —	symmetric 2-group	sylleptic 3-group	$\cdots$	sylleptic ∞-group
4	— " —	— " —	symmetric 3-group	$\cdots$	?? ∞-group
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	— " —	$\cdots$	connective spectrum

(De)categorification

$k \setminus n$	0	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	3-group	$\cdots$	∞-group
2	abelian group	braided 2-group	braided 3-group	$\cdots$	braided ∞-group
3	— " —	symmetric 2-group	sylleptic 3-group	$\cdots$	sylleptic ∞-group
4	— " —	— " —	symmetric 3-group	$\cdots$	?? ∞-group
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	— " —	$\cdots$	connective spectrum

discrete categorification $\mathrm{Disc} : (n,k)\mathrm{GType} \to (n+1,k)\mathrm{GType}$ $\longrightarrow$

$$\langle G, B^k G\rangle \mapsto \langle G, B^k G\rangle$$

decategorification $\mathrm{Decat} : (n,k)\mathrm{GType} \to (n-1,k)\mathrm{GType}$ $\longleftarrow$

$$\langle G, B^k G\rangle \mapsto \langle \|G\|_{n-1}, \|B^k G\|_{n+k-1}\rangle$$

$\mathrm{Decat} \dashv \mathrm{Disc}$ and $\mathrm{Decat} \circ \mathrm{Disc} = \mathrm{id}$

(De)looping

$k \setminus n$	0	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	— " —	$\cdots$	connective spectrum

looping $\Omega : (n, k)\text{GType} \to (n-1, k+1)\text{GType}$ $\quad\swarrow$

$$\langle G, B^k G \rangle \mapsto \langle \Omega G, B^k G \langle k \rangle \rangle$$

delooping $\mathrm{B} : (n, k)\text{GType} \to (n+1, k-1)\text{GType}$ $\quad\nearrow$

$$\langle G, B^k G \rangle \mapsto \langle \Omega^{k-1} B^k G, B^k G \rangle$$

$\mathrm{B} \dashv \Omega$ and $\Omega \circ \mathrm{B} = \mathrm{id}$

Stabilization

$k \setminus n$	0	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	3-group	$\cdots$	∞-group
2	abelian group	braided 2-group	braided 3-group	$\cdots$	braided ∞-group
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4	— " —	— " —	symmetric 3-group	$\cdots$	?? ∞-group
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	— " —	$\cdots$	connective spectrum

forgetting $F : (n,k)\mathrm{GType} \to (n,k-1)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle G, \Omega B^k G\rangle$ $\uparrow$

stabilization $S : (n,k)\mathrm{GType} \to (n,k+1)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle \|\Omega^{k+1}\Sigma B^k G\|_n, \|\Sigma B^k G\|_{n+k+1}\rangle$ $\downarrow$

$$S \dashv F$$

Formalization in Lean

Theorem

If $G, H : (n, k)\mathrm{GType}$ *then* $\hom_{(n,k)}(G, H) := B^k G \to_{\mathrm{pt}} B^k H$ *is* n*-truncated. Hence* $(n, k)\mathrm{GType}$ *is* $(n + 1)$*-truncated.*

Theorem (Set-level groups)

We have the following equivalences of categories:

$$\begin{aligned} (0,0)\mathrm{GType} &\simeq \mathrm{Set}_{\mathrm{pt}}; \\ (0,1)\mathrm{GType} &\simeq \mathrm{Group}; \\ (0,k)\mathrm{GType} &\simeq \mathrm{AbGroup} \qquad (\textit{for } k \geq 2). \end{aligned}$$

Theorem (Stabilization)

If $k \geq n + 2$*, then* $S : (n, k)\mathrm{GType} \to (n, k + 1)\mathrm{GType}$ *is an equivalence, and any* $G : (n, k)\mathrm{GType}$ *is an infinite loop space.*

Examples

- The integers has delooping $B\mathbb{Z} = \mathbb{S}^1$.
- The free 1-group on a set X has delooping $BF_X = \|\Sigma(X+1)\|_1$.
- The automorphism group of $a : A$ is $\mathrm{Aut}\, a :\equiv (a = a)$, with delooping

$$\mathrm{BAut}\, a :\equiv \mathsf{im}(a : 1 \to A) = (x : A) \times \|a = x\|_{-1}.$$

- The fundamental n-group of (A, a) is $\Pi_n(A, a) :\equiv \|a = a\|_{n-1}$, the decategorification of the automorphism group.
- The symmetric groups $S_n :\equiv \mathrm{Aut}(\mathsf{fin}_n)$ has as delooping the n-element sets $BS_n = (A : \mathrm{Type}) \times \|A \simeq \mathsf{fin}_n\|_{-1}$.

$\vdots$

Actions

- A G-action on $a : A$ is a homomorphism $G \to \operatorname{Aut} a$, or equivalently, a pointed map $BG \to_{\mathrm{pt}} (A, a)$
- A G-type is a function $X : BG \to \mathrm{Type}$, that is, an action on a type.
- The homotopy fixed points or invariants are

$$X^{hG} :\equiv (z : BG) \to X(z).$$

- The homotopy orbit space or coinvariants are

$$X /\!\!/ G :\equiv (z : BG) \times X(z).$$

- The stabilizer of $x : X(\mathrm{pt})$ is $G_x :\equiv \mathrm{Aut}(\langle \mathrm{pt}, x \rangle : X /\!\!/ G)$
- The orbit of $x : X(\mathrm{pt})$ is

$$G \cdot x :\equiv (y : X(\mathrm{pt})) \times \| \langle \mathrm{pt}, x \rangle = \langle \mathrm{pt}, y \rangle \|_{-1}.$$

Theorem (Orbit-Stabilizer Theorem)

For $x : X(\mathrm{pt})$ we have $G /\!\!/ G_x \simeq G \cdot x$.

Concluding Remarks

- Homotopy type theory gives a convenient language for higher group theory.
- We can do higher group theory. There is more in the paper.
- Future work: prove that more entries of the periodic table are equivalent to the classical definition.