Eilenberg-MacLane spaces in Homotopy Type Theory

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Models of Type Theory

There are models of *type theory* in various abstract frameworks for *homotopy theory*.

Examples:

- Quillen model categories [Awodey, Warren, 2009];
- Simplicial sets [Streicher, 2011];
- Cubical sets [Bezem, Coquand, Huber, 2014];
- . . . and many more.

Synthetic Homotopy Theory

This leads to a new program, *Synthetic Homotopy Theory*:

Study types in type theory as spaces in homotopy theory.

This gives a more general and constructive treatment of homotopy theory which is easier to verify formally in a computer proof assistant.

- The main theorem in this talk has been fully formalized.

I work in *Homotopy Type Theory* (HoTT): dependent type theory with *univalence* and *higher inductive types* [Homotopy Type Theory, 2013].

As motivating example I will concentrate on *Eilenberg-MacLane spaces*.

Homotopy Type Theory

Homotopy Type Theory combines Type Theory with Homotopy Theory.

	Type Theory	**Logic**	**Homotopy Theory**
A	Type	Formula	Space*
$a : A$	Term/Element	Proof	Point
$A \times B$	Product Type	Conjunction	Binary Product sp.
$A \to B$	Function Type	Implication	Mapping space
$P : A \to \mathrm{Type}$	Dependent Type	Predicate	Fibration
$\Sigma(x : A).\ P(x)$	Sigma Type	Ex. Quantifier	Total space
$\Pi(x : A).\ P(x)$	Dep. Fn. Type	Un. Quantifier	Product space
$a =_A b$	Identity Type	Equality	Path space

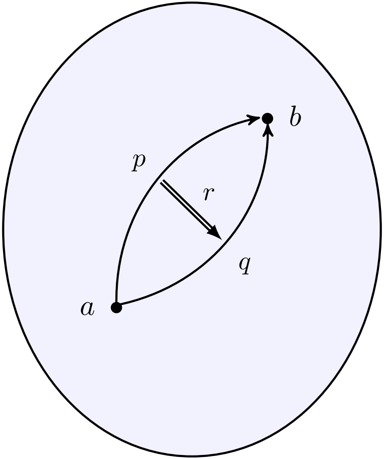
I will use these notions interchangably.

Types as spaces

A type A can have

- points $a, b : A$
- paths $p, q : a = b$
- paths between paths $r : p = q$

$$\vdots$$



Identity Type

Different ways to think about the identity type:

- **Type theory**: The identity type is generated by reflexivity: $\mathrm{refl}_a : a =_A a$.
- **Logic**: Equality is the least (free) reflexive relation.
- **Homotopy theory**: The path space with one point fixed is contractible.

(This does not mean every proof of equality is reflexivity)

Path Induction

This is made precise by *path induction*:

- If $C : \Pi(x : A).\ a = x \to \mathrm{Type}$,
- to prove/construct an element of $\Pi(x : A).\Pi(p : a = x).\ C(x, p)$
- it is sufficient to prove/construct an element of $C(a, \mathrm{refl}_a)$

Example Symmetry of equality (invertibility of paths)

$$\Pi(A : \mathrm{Type}).\ \Pi(a\ b : A).\ a = b \to b = a.$$

Proof. Suppose A is a type and $a : A$. We need to prove $\Pi(b : A).\ a = b \to b = a$.
We apply path induction, in which case we need to prove $a = a$, which is true by refl_a.

Identity Type (2)

We can also look at the identity type in a type-oriented way:

$$
\begin{array}{rcll}
(a,b) =_{A\times B} (a',b') & \text{is} & (a =_A a') \times (b =_B b') & \\
f = g & \text{is} & \Pi x.\ f(x) = g(x) & \text{(function extensionality)} \\
A =_{\mathrm{Type}} B & \text{is} & A \simeq B & \text{(univalence, Voevodsky)}
\end{array}
$$

This is done in *cubical type theory*.

Truncated Types

Some types are *truncated*, which means there are all higher paths are trivial.

A type A is *contractible* ((-2)-type) if it has exactly one element, if

$$\Sigma(x : A).\ \Pi(y : A).\ x = y.$$

A type A is a *proposition* ((-1)-type) if it has at most one element, if

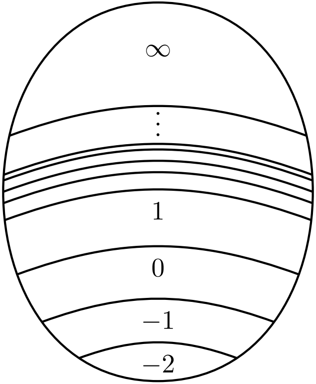
$$\Pi(x\ y : A).\ x = y.$$

In either of the above cases, all (higher) paths in A are trivial.

A is a *set* (0-type) if for all $x\ y : A$ the type $x = y$ is a proposition.

A is an $(n+1)$-type if for all $x\ y : A$ the type $x = y$ is an n-type.

Truncated Types



Truncation

Given A, we can form the n-*truncation* $\|A\|_n$.

$\|A\|_n$ is the "best approximation" of A which is n-truncated.

If X is n-truncated, we get the following universal property:

$$\begin{tikzcd} A \arrow[d, "|-|_n"'] \arrow[dr, "\forall"] & \\ \|A\|_n \arrow[r, dashed, "\exists!"'] & X \end{tikzcd}$$

Higher Inductive Types

In Type Theory there are *inductive types*, in which you specify its points.

Examples. $\mathbb{N}$ is generated by 0 and succ
$A + B$ is generated by either $a : A$ or $b : B$
$a =_A (-)$ is generated by $\mathrm{refl}_a : a =_A a$

In homotopy theory we can build cell complexes inductively.

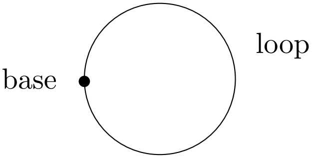
In HoTT we can combine these into *higher inductive types* [Shulman, Lumsdaine, 2012].

The circle

Example. The circle $\mathbb{S}^1$

$\texttt{HIT}\ \mathbb{S}^1 :=$

- $\mathrm{base} : \mathbb{S}^1$
- $\mathrm{loop} : \mathrm{base} = \mathrm{base}$

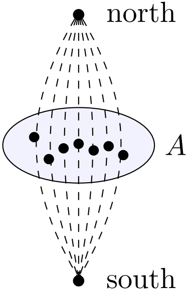


Using univalence, we can prove $\mathrm{loop} \neq \mathrm{refl}$.

Recursion Principle. To define $f : \mathbb{S}^1 \to A$ we need to define $a : A$ and $p : a = a$.

The suspension

Example. The suspension ΣA



$\texttt{HIT}\ \Sigma A :=$

- $\mathrm{north}, \mathrm{south} : \Sigma A$
- $\mathrm{merid} : A \to (\mathrm{north} = \mathrm{south})$

Remark. $\mathbb{S}^1 \simeq \Sigma \mathbf{2}$

Definition. We can now define the n-spheres by $\mathbb{S}^{n+1} := \Sigma \mathbb{S}^n$ and $\mathbb{S}^0 := \mathbf{2}$

Homotopy Groups

In algebraic topology, we look for algebraic invariants of spaces, like the *homotopy groups*.

Traditionally: $\pi_n(A, a_0) = \{f : \mathbb{S}^n \to A \mid f \text{ preserves basepoints}\}/\sim$.

In HoTT $\pi_n(A, a_0) = \|\mathbb{S}^n \to^* A\|_0$ where we use $\to^*$ for basepoint preserving maps.

Alternative characterization: $\pi_n(A, a_0) = \|\Omega^n(A, a_0)\|_0$ where $\Omega(A, a_0) = (a_0 = a_0, \mathrm{refl}_{a_0})$.

These are groups for $n \geq 1$ (abelian for $n \geq 2$).

Connectedness and truncatedness

If X is n-truncated then $\pi_k(X) = 0$ for all $k > n$.

The converse is not true in general.

Definition. A type A is *n-connected* if $\|A\|_n$ is contractible.

Remark. (-1)-connected: merely inhabited;
0-connected: path-connected;
1-connected: simply connected.

X is n-connected if and only if $\pi_k(X) = 0$ for all $k \leq n$.

If X is n-connected, then ΣX is $(n+1)$-connected.

Thus the n-sphere $\mathbb{S}^n$ is $(n-1)$-connected.

Homotopy Groups of spheres

	$\mathbb{S}^0$	$\mathbb{S}^1$	$\mathbb{S}^2$	$\mathbb{S}^3$	$\mathbb{S}^4$	$\mathbb{S}^5$	$\mathbb{S}^6$	$\mathbb{S}^7$	$\mathbb{S}^8$
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{11}	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$
π_{12}	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0
π_{13}	0	0	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0

Homotopy Groups of spheres

	$\mathbb{S}^0$	$\mathbb{S}^1$	$\mathbb{S}^2$	$\mathbb{S}^3$	$\mathbb{S}^4$	$\mathbb{S}^5$	$\mathbb{S}^6$	$\mathbb{S}^7$	$\mathbb{S}^8$
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{11}	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$
π_{12}	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0
π_{13}	0	0	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0

Eilenberg MacLane spaces

Question. Can we construct spaces with simple homotopy groups?

In classical homotopy theory, the *Eilenberg MacLane space* $K(G,n)$ is the *unique* space such that

$$\pi_n(K(G,n)) = \begin{cases} G & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

We have already seen one example $K(\mathbb{Z},1) = \mathbb{S}^1$.

Eilenberg-MacLane spaces classify homology and cohomology.

These can be constructed in HoTT [Licata, Finster, 2014].

We write $\mathrm{Type}_*^{=n}$ for the universe of pointed $(n-1)$-connected n-truncated types. We want to construct $K(G,n) : \mathrm{Type}_*^{=n}$.

Eilenberg MacLane space $K(G,1)$

For $n = 1$, suppose G is a group.

HIT $\widetilde{K}(G,1) :=$

- $\star : \widetilde{K}(G,1)$
- $\mathrm{pth} : G \to (\star = \star)$
- $\text{pth-mul} : \Pi(g\ h : G).\ \mathrm{pth}(gh) = \mathrm{pth}(g) \cdot \mathrm{pth}(h)$

$\widetilde{K}(G,1)$ is not quite an Eilenberg-MacLane space; it has nontrivial higher structure.

$$K(G,1) := \|\widetilde{K}(G,1)\|_1$$

$K(G,1)$ is 0-connected, 1-truncated (it lives in $\mathrm{Type}_*^{=1}$) and one can show that $\pi_1(K(G,1)) = G$.

Eilenberg MacLane space $K(G, n)$

Suppose $n \geq 1$. We want to construct $K(G, n+1)$ out of $K(G, n)$. This only works if G is abelian.

Definition. $K(G, n+1) :\equiv \|\Sigma K(G, n)\|_{n+1}$

Now $K(G, n+1)$ is indeed n-connected and $(n+1)$-truncated (it lives in $\mathrm{Type}_*^{=n+1}$). We can show that $\Omega K(G, n+1) = K(G, n)$ (if G is abelian). Hence

$$\begin{aligned} \Omega^{n+1} K(G, n+1) &= \Omega^n \Omega K(G, n+1) \\ &= \Omega^n K(G, n) \\ &= G \end{aligned}$$

So $K(G, n)$ has the right homotopy groups.

Main result

Theorem. Any $X : \mathrm{Type}_*^{=n}$ is equivalent to $K(\pi_n(X), n)$.
Moreover, $K(-, n)$, interpreted as a functor from $\mathrm{AbGrp} \to \mathrm{Type}_*^{=n}$ is an equivalence of categories for $n \geq 2$.
For $n = 1$ it is an equivalence of categories $\mathrm{Grp} \to \mathrm{Type}_*^{=1}$.

This means that not only every $X : \mathrm{Type}_*^{=n}$ is an Eilenberg-MacLane space, but also any map $f : X \to Y$ is given by the action of a unique group homomorphism on Eilenberg MacLane spaces.

Special case: uniqueness of $K(G, 1)$

As a special case we show: if $(X, x_0) : \mathrm{Type}_*^{=1}$ and we have a group isomorphism $e : G \simeq \pi_1(X)$ then $K(G, 1) \simeq X$.

$\texttt{HIT}\ \widetilde{K}(G, 1) :=$

- $\star : \widetilde{K}(G, 1)$
- $\mathrm{pth} : G \to \star = \star$
- $\text{pth-mul} : \Pi(g\ h : G).\ \mathrm{pth}(gh) = \mathrm{pth}(g) \cdot \mathrm{pth}(h)$

$K(G, 1) = \|\widetilde{K}(G, 1)\|_1$

Recursion Principle. To define $f : K(G, 1) \to A$ for a 1-type A we need $a : A$ and $p : G \to a = a$ such that $p(gh) = p(g) \cdot p(h)$.

We define a map $f : K(G, 1) \to X$ by sending $\star$ to x_0, $\mathrm{pth}(g)$ to $e(g)$, viewed as element of ΩX, and $e(gh) = e(g)e(h)$ because e is a group homomorphism. Is f an equivalence?

Special case: uniqueness of $K(G,1)$

f induces an isomorphism on $\pi_k(K(G,1)) \to \pi_k(X)$ for all k (trivially for $k \neq 1$).
Such an f is called a *weak equivalence*.
For $k = 1$, we use that the following triangle commutes:

$$\begin{tikzcd}
\pi_1(K(G,1)) \arrow[rr, "\pi_1(f)"] \arrow[dr, equal] & & \pi_1(X) \arrow[dl, equal, "e"] \\
& G &
\end{tikzcd}$$

Theorem. (Whitehead) If $g : A \to B$ is a weak equivalence, A and B are n-types for some n, then f is an equivalence.

Hence $f : K(G,1) \to X$ is an equivalence.

Lean

This result is formally proven in the proof assistant *Lean*.

Lean is a new open source proof assistant with support for HoTT, similar to Coq and Agda.

Lean implements dependent type theory with a hierarchy of (non-cumulative) universes and inductive types (à la Dybjer, with constructors and recursors).

The kernel is smaller and simpler than those of Coq and Agda.

Lean has two modes: a standard mode for classical and constructive reasoning and a HoTT mode for Homotopy Type Theory.

HoTT library

The Lean HoTT library contains an extensive collection of basic concepts, and the following results were formalized:

- The Freudenthal Suspension Theorem
- The Hopf fibration
- The long exact sequence of homotopy groups
- The Seifert-van Kampen theorem
- The adjunction between the smash product and pointed maps
- Eilenberg MacLane spaces

Currently I'm working in a group project to formalize spectral sequences in Lean.

Code snippets

```
definition KG1_map {G : Group} {X : Type*} (e : G → Ω X)
  (r : Πg h, e (g * h) = e g · e h) [is_conn 0 X] [is_trunc 1 X]
  : K G 1 → X :=
begin
  intro x, induction x using EM.elim,
  { exact Point X },
  { exact e g },
  { exact r g h }
end

definition Grp_equivalence : Grp ≃c cType*[1] :=
equivalence.mk EM1_cfunctor is_equivalence_EM1_cfunctor

definition AbGrp_equivalence (n : ℕ) : AbGrp ≃c cType*[n+2] :=
equivalence.mk (EM_cfunctor (n+2)) (is_equivalence_EM_cfunctor n)
```

Conclusion

Advantages of Synthetic homotopy theory:

- More general
  - There are multiple models of HoTT;
- The homotopy theoretic notions are primitives in type theory
  - We don't have to talk about topology, continuity, ...
- Novel ways of reasoning
  - Path induction, homotopy invariance;
- Constructive (but not anti-classical)
  - Has computational interpretation;
- Possible to verify formally in practice
  - Proof fully formalized in Lean.

Thank you

The Lean HoTT library is available at:

https://github.com/leanprover/lean2/blob/master/hott/hott.md