The Lean-HoTT library

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Lean HoTT library

It is the Lean standard library without Prop, but with univalence and 2 primitive HITs: the n-truncation and quotients (which are interdefinable with pushouts)

Homotopy Theory in Lean

Theorems which were already proven last year:

- Loop space of the circle
- Connectedness of suspensions
- The real and complex hopf fibrations
- The Freudenthal suspension theorem
- The long exact sequence of homotopy groups

Formalized parts of chapter 8:

	1	2	3	4	5	6	7	8	9	10
last year	+	+	+	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	-	-	$\frac{1}{2}$	-
now	+	+	+	+	+	$\frac{3}{4}$	+	+	+	$\frac{1}{4}$

Homotopy Theory in Lean

	1	2	3	4	5	6	7	8	9	10
last year	+	+	+	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	-	-	$\frac{1}{2}$	-
now	+	+	+	+	+	$\frac{3}{4}$	+	+	+	$\frac{1}{4}$

New Theorems:

- $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ (for $n \geq 1$) and $\pi_n(\mathbb{S}^3) = \pi_n(\mathbb{S}^2)$ (for $n \geq 3$) (j.w.w. Ulrik Buchholtz)
- The Seifert-van Kampen theorem: (with basepoints)
  - "the fundamental groupoid of a pushout is weakly equivalent (as categories) to the pushout of the fundamental groupoids."
- Whitehead's principle:
  - "A weak equivalence between truncated types is an equivalence."
- Eilenberg-MacLane spaces:
  - We can construct pointed types $K(G, n)$ which are n-truncated and $(n-1)$-connected and n-th homotopy group G, ...

- Eilenberg-MacLane spaces:

  [Let $\mathrm{Type}_*^{=n}$ be the universe of n-truncated $(n-1)$-connected types]

  - $\ldots$ then $K(-,1)$ induces an equivalence between the categories $\mathrm{Grp} \to \mathrm{Type}_*^{=1}$
  - and for $n \geq 2$ the functor $K(-,n)$ induces an equivalence $\mathrm{AbGrp} \to \mathrm{Type}_*^{=n}$
    (j.w.w. Ulrik Buchholtz and Egbert Rijke)

Homotopy Theory in Lean

- Properties about the smash product: (main part of my talk)

$$((-) \wedge B) \dashv (B \to^* (-)) \quad \text{(natural in } B\text{)}$$

$$A \wedge (B \wedge C) \simeq^* (A \wedge B) \wedge C \quad \text{(natural in } A\text{, } B \text{ and } C\text{)}$$

  (j.w.w. Robin Adams, Mark Bezem, Ulrik Buchholtz, Stefano Piceghello, Egbert Rijke)

- [In Progress] Formalization of spectral sequences, in particular the Serre Spectral Sequence.

  (j.w.w. Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke, Mike Schulman)

Spectral Sequences (as described by Mike Shulman)

Definition A (homologically indexed) *spectral sequence* consists of

- A family $(E^r_{p,q})$ of R-modules (or objects in an abelian category) for $p, q : \mathbb{Z}$ and $r \geq 2$. For a fixed r this gives the r*-page* of the spectral sequence.
- (homo)morphisms $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ which are called *differentials*.
- isomorphisms $\alpha^r_{p,q} : H_{p,q}(E^r) \simeq E^{r+1}_{p,q}$ where $H_{p,q}(E^r) = \ker(d^r_{p,q}) / \operatorname{im}(d^r_{p+r,q-r+1})$.

We build these in the following way:

- Given an *iterated fibration sequence*;
- We construct an *exact couple*;
- We iteratively build a *derived exact couple*;
- These give a spectral sequence, which under certain conditions converges.

Iterated fibration sequence

Given a sequence of maps

$$Y_T \xrightarrow{f_T} Y_{T-1} \xrightarrow{f_{T-1}} Y_{T-2} \xrightarrow{f_{T-2}} \cdots$$

Let $X_s :\equiv \mathrm{fib}_{f_s}$ We build the iterated fibration sequence:

$$\begin{aligned} X_T \to & Y_T \to Y_{T-1} \\ X_{T-1} \to & Y_{T-1} \to Y_{T-2} \\ X_{T-2} \to & Y_{T-2} \to Y_{T-3} \\ & \vdots \end{aligned}$$

We want to compute $\pi_n(Y_T)$ from the homotopy groups of X_s.

We assume that for every n there is an R that such that $\pi_n(X_s) = \pi_n(Y_s) = 0$ for $s \leq R$.

Exact couple

Define $E^2_{p,q} :\equiv \pi_{p+q}(X_q)$ and $D^2_{p,q} :\equiv \pi_{p+q}(Y_q)$. The long exact sequence of homotopy groups gives

$$\cdots \to \pi_n(X_s) \to \pi_n(Y_s) \to \pi_n(Y_{s-1}) \to \pi_{n-1}(X_s) \to \cdots .$$

which gives the *exact couple*

$$\begin{tikzcd} D^2 \arrow[rr, "i"] & & D^2 \arrow[dl, "j"] \\ & E^2 \arrow[ul, "k"] & \end{tikzcd}$$

Derived Exact couple

From an exact couple

$$\begin{tikzcd} D^2 \arrow[rr, "i^2"] & & D^2 \arrow[dl, "j^2"] \\ & E^2 \arrow[ul, "k^2"] & \end{tikzcd}$$

we build a *derived exact couple*

$$\begin{tikzcd} D^3 \arrow[rr, "i^3"] & & D^3 \arrow[dl, "j^3"] \\ & E^3 \arrow[ul, "k^3"] & \end{tikzcd}$$

with $E^3_{p,q} = H_{p,q}(E^2)$ with differential $d^2 :\equiv j^2 k^2 : E^2 \to E^2$.

Spectral Sequence

We iterate this process, and make construct the exact couple $(E^{r+1}, D^{r+1}, i^{r+1}, j^{r+1}, k^{r+1})$ as the derived couple of $(E^r, D^r, i^r, j^r, k^r)$.

Then $(E^r, d^r)_r$ forms an spectral sequence. For given p, q the sequence $(E^r_{p,q}$ is eventually constant, and we define the eventual value as $E^\infty_{p,q}$.

Convergence Theorem

Recall: We assume that for every n there is an R that such that $\pi_n(X_s) = \pi_n(Y_s) = 0$ for $s \leq R$.

Theorem There are abelian groups $B_{n,s}$ with $B_{n,T} = \pi_n(Y)$ and finite iterated extensions (short exact sequences)

$$\begin{aligned}
E^\infty_{n-T,T} \to& B_{n,T} \to B_{n,T-1} \\
&\vdots \\
E^\infty_{n-s,s} \to& B_{n,s} \to B_{n,s-1} \\
E^\infty_{n-s+1,s-1} \to& B_{n,s-1} \to B_{n,s-2} \\
&\vdots \\
E^\infty_{n-R,R} \to& B_{n,R} \to 0
\end{aligned}$$

This is denoted $\pi_{p+q}(X_q) \Rightarrow \pi_{p+q}(Y_T)$.

Serre Spectral Sequence

Theorem

Given a pointed map $f : X \to B$ with fiber F where B is simply connected. For a spectrum Y we get

$$H^p(B; H^q(F; Y)) \Rightarrow H^{p+q}(X; Y).$$

Here $H^n(X; Y) :\equiv \|X \to Y_n\|_0$ and $H^n(X; G) :\equiv H^n(X; K(G, -))$ for an abelian group G.

Progress (globally)

We have:

- Eilenberg-MacLane spaces
- basic theory of spectra (LES of homotopy groups)
- cohomology theory satisfies Eilenberg-Steenrod axioms
- Basic algebraic constructions
- Long exact sequence of homotopy groups

To do:

- Derive an exact couple (in progress)
- The Convergence Theorem
- Spectrification and other constructions on spectra
- Cohomology with local coefficients