The internals of Lean

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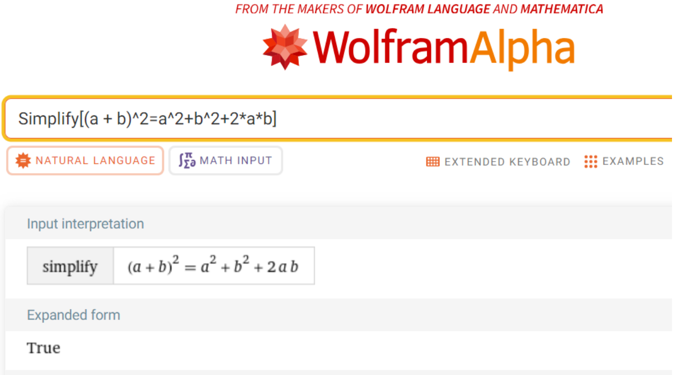
Questions discussed

- What is the logic of Lean?
- How does Lean check a proof?
- Can we trust proofs checked by Lean?

Computer algebra systems and proof assistants



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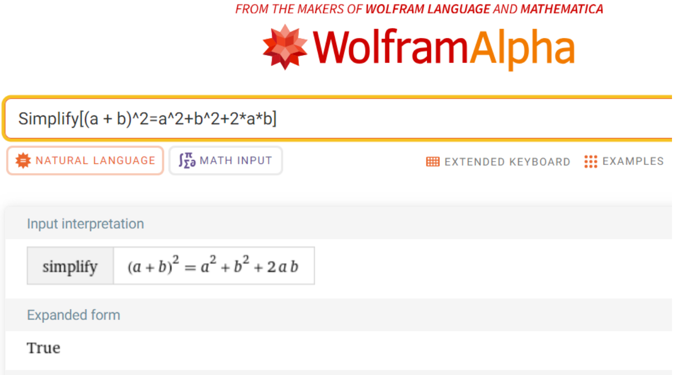
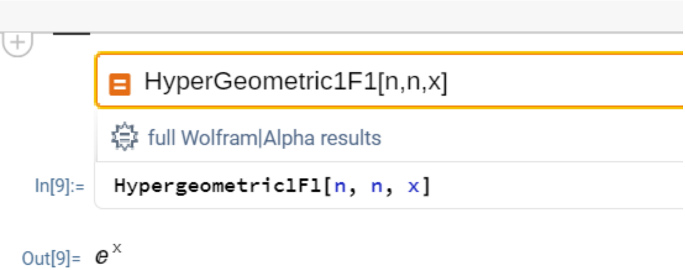



```
example (a b : ℂ) :
  (a + b)^2 = a^2 + b^2 + 2*a*b := by ring
```

Computer algebra systems and proof assistants



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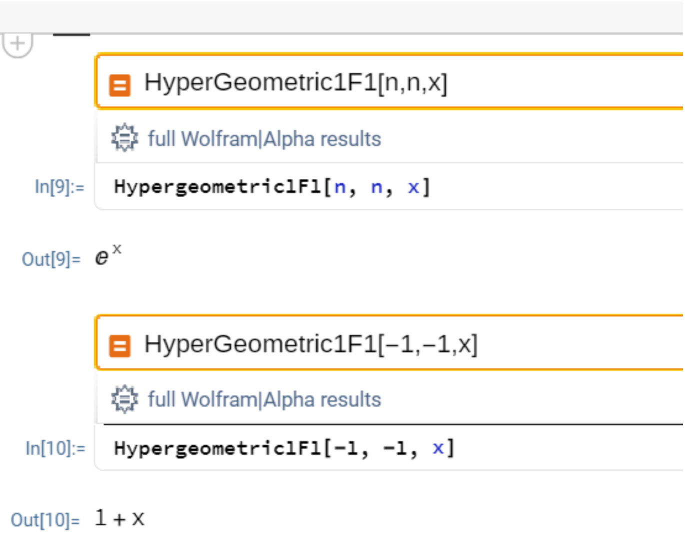

Computer algebra systems and proof assistants

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Proof Assistant User writes a statement and proof, the program will check it.

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Automated Theorem Prover User writes a statement, the program will find a proof or fail.

Logic of a proof assistant

A proof assistant implements a particular logic in which the proofs are checked.

Set theory Mizar, Metamath*

Simple type theory HOL Light, Isabelle*

Dependent type theory Lean, Coq

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You don't need to know the logic to start doing mathematics with a proof assistant

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- $a + b$ (for a, b in some ring R) means addition in R
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More complicated expressions:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Type theory

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$$\pi : \mathbb{R}$$
$$i : \mathbb{C}$$
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In set theory this is harder, it's "too flexible".

Warning

We have $3 : \mathbb{N}$ and $3 : \mathbb{R}$.

In type theory, these two 3's are not the same object.

(Of course, canonical inclusion $\mathbb{N} \hookrightarrow \mathbb{R}$ sends the former to the latter.)

In Lean, you can write `(3 : ℕ)` or `(3 : ℝ)` to force an expression to have a particular type.

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inductive ℕ where
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Definitional equality There is a notion of computation: $2 + 2 \equiv 4$, $(x, y).1 \equiv x$.

`rfl` can prove $a = b$ precisely when a and b are definitionally equal.

Dependent type theory

There are some details in Lean's type theory that are a bit complicated:

Useful to learn let-expressions, quotients, axiom of choice

A bit obscure universe levels, proof irrelevance, propositional extensionality

Very obscure impredicative `Prop`, subsingleton elimination, $\alpha\beta\delta\eta\zeta\iota$-conversion

Soundness

Is Lean's logic sound?

Short answer: Yes

Soundness

Is Lean's logic sound?

Yes, modulo issues with Gödels incompleteness theorem

Soundness

Is Lean's logic sound?

It is weaker than ZFC $+ \omega$ inaccessible cardinals

Lean

```
@[simp]
theorem integral_sin : ∫ x in a..b, sin x = cos a - cos b := by
  rw [integral_deriv_eq_sub' fun x => -cos x]
  · ring
  · norm_num
  · simp only [differentiableAt_neg_iff, differentiableAt_cos, implies_true]
  · exact continuousOn_sin
```

Processing a proof

What happens after writing a proof?

- Parsing (interpreting notation)
- Elaboration (figure out implicit information)
- Tactic execution
- Kernel checking

Elaboration

```
theorem add_comm {G : Type*} [AddCommMagma G]
  (a : G) (b : G) :
  a + b = b + a

example (a b c : ℝ) : a * b + c = c + a * b := by
  exact add_comm (a * b) c
```

- Lean figures out that `(G := ℝ)` from context (by looking at the type of `a`, `b` and `c`)
- Lean has a database of types where addition commutes, and looks up to see that it is true for $\mathbb{R}$ (*type-class inference*)

Tactic execution

Tactics can be any program that construct part of the proof.

Simple tactics that do 1 step in a proof: `intro`, `apply`, `have`, `rw`;
Domain-specific automation: `ring`, `linarith`
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Tactics produce a proof term.
(usually giant, unreadable for humans)

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The kernel is a (relatively) small part of Lean, and it is the trusted codebase.

To trust that Lean only accepts true theorems, you only have to trust the kernel. You do not have to trust tactics.

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To verify a formalization of non-malicious user:

- check the theorem statement
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If you are really paranoid:

- trust consistency of ZFC $+ \omega$ inaccessibles
- trust the compiler that compiled the type checker down to machine code
- trust that your hardware follows specifications
- trust that no cosmic rays interfered with your hard drive

Extensibility

Demo You can declare your own notation

```
notation3 "∫ "(...)" in "a".."b",
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elab "my_assumption" : tactic ⇒ do
let target ← getMainTarget
for ldecl in ← getLCtx do
  if ldecl.isImplementationDetail then continue
  if ← isDefEq ldecl.type target then
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    return
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In fact, almost every part of Lean (parsing, elaboration, tactics, compilation) are written in Lean

Conclusions

- Type theory is a useful logic for formalization;
- You can trust Lean formalizations;
- Lean is very extensible.