Formalizing mathematics in Lean

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November 9, 2021

Proof assistants

Proof assistants are programs that can check the validity of a proof if that proof is written in a language it can understand.

In CS, this is used to prove that software or hardware has no bugs.

- CompCert: A formally-verified C compiler.

In math, this is used to prove deep mathematical theorems.

- The Kepler conjecture: A formally verified proof of a four centuries old problem
  - Reviewers were unable to check the proof given on paper.

Verifying Math/CS: similarities

Verifying math is similar to verifying software.

- You can use proof assistants for either purpose.
- You have to carefully write down the statement, including all assumptions.
- Automation of routine steps is immensely helpful.

Verifying Math/CS: differences

Verifying math is different from verifying software.
Math ...

- recursively builds on itself
- is very interconnected

In CS ...

- Often definitions are much larger
- Proofs often need to check many different cases

Lean

Lean is a proof assistant developed by Leonardo de Moura at Microsoft Research.

It has a type theory with dependent types, very similar to Coq.

```
def nextPrime (n : ℕ) : { m : ℕ //
  prime m ∧ ∀ (k : ℕ), prime k → n < k ↔ m ≤ k }
```

This can be seen as

- a program that computes the smallest prime larger than the input;
- a proof that there are infinitely many primes.

mathlib

mathlib is the mathematical library of Lean.

It contains many areas of mathematics in a single library.

It has 440k lines of code + 100k lines of documentation/comments.

There are 180+ contributors to mathlib. All contributions are reviewed by at least one of the maintainers.

There is a highly active community on `leanprover.zulipchat.com` with discussions, people helping each other, people teaching newcomers.

Demo

Let's prove a basic exercise in topology:

Lemma

If $f : X \to Y$ is a map between two topological spaces that is continuous at every $x \in X$, then f is a continuous.

Recall:

Definition

f is continuous if $f^{-1}(U)$ is open (in X) for every open $U \subseteq Y$.
A is a neighborhood of x if it is a superset of an open set containing x.
Notation: $A \in \mathcal{N}_x$.
f is continuous at x if $f^{-1}(A) \in \mathcal{N}_x$ for every neighborhood $A \in \mathcal{N}_{f(x)}$.

Demo

Lemma

If $f : X \to Y$ is a map between two topological spaces that is continuous at every $x \in X$, then f is a continuous.

Proof.

Let $U \subseteq Y$ be open.
To show that $f^{-1}(U)$ is open, it is sufficient to show that $f^{-1}(U)$ is a neighborhood of every point $x \in f^{-1}(U)$.
Therefore take x in $f^{-1}(U)$. Then U is a neighborhood of $f(x)$, hence $f^{-1}(U)$ is a neighborhood of x. Therefore $f^{-1}(U)$ is open. $\square$

Topics in mathlib: algebra

```lean
/-- **Abel-Ruffini Theorem**: not every polynomial root
  is expressible using radicals. -/
theorem exists_not_solvable_by_rad :
  ∃ x : ℂ, is_algebraic ℚ x ∧ ¬ is_solvable_by_rad ℚ x
```

The proof specifically shows that the roots of $x^5 - 4x + 2$ are not expressible by radicals.

```
/-- Fundamental theorem of calculus, part 2 -/
theorem integral_deriv_eq_sub
  (hderiv : ∀ x ∈ interval a b, differentiable_at ℝ f x)
  (hint : interval_integrable (deriv f) volume a b) :
  ∫ y in a..b, deriv f y = f b - f a
```

Topics in mathlib: combinatorics

There does not exist a partition of a hypercube in dimension $n \geq 3$ into finitely many smaller cubes of different sizes.

```
/-- **Dissection of Cubes**: A cube cannot be cubed. -/
theorem cannot_cube_a_cube :
  ∀{n : ℕ}, n ≥ 3 →
  ∀{ι : Type} [fintype ι] {cs : ι → cube n},
  2 ≤ #ι →
  pairwise (disjoint on (cube.to_set ∘ cs)) →
  (⋃(i : ι), (cs i).to_set) = unit_cube.to_set →
  injective (cube.width ∘ cs) →
  false
```

Goals of mathlib

The goal of mathlib is to be a general-purpose library for all areas of mathematics.

It is decentralized: every contributor comes with their own plans and goals.

One unified goal is to have a full undergraduate math curriculum by next year.

Projects

- Definition of perfectoid spaces (Buzzard, Commelin, Massot)
- Independence of the Continuum hypothesis (Han, van Doorn)
- Witt vectors (Commelin, Lewis)
- Huang's sensitivity theorem (Barton, Commelin, Han, Hughes, Lewis, Massot)
- Finiteness of the class group of a global field (Baanen, Dahmen, Narayanan, Nuccio)
- Liquid Tensor Experiment (Commelin et al)
- Sphere Eversion and convex integration (Massot, Nash, van Doorn)

Tools

We have various tools to help develop and maintain mathlib.

- Automatically generated documentation pages;
- Tactics for general-purpose or domain-specific automation
  - `suggest` searches the library for an applicable lemma.
  - `simp`: general-purpose simplifier.
  - `abel`, `ring`, `linarith`, `omega`, `continuity`: domain-specific automation.
  - `tidy`, `finish`, `solve_by_elim`: general purpose automation.

Tools: simps

Consider:

```
def yoneda C ⥤ (Cᵒᵖ ⥤ Type v₁) :=
{ obj := λ X,
  { obj := λ Y, unop Y ⟶ X,
    map := λ Y Y' f g, f.unop ≫ g },
  map := λ X X' f, { app := λ Y g, g ≫ f } }

@[simp] lemma obj_obj (X : C) (Y : Cᵒᵖ) :
  (yoneda.obj X).obj Y = (unop Y ⟶ X) := rfl
@[simp] lemma obj_map (X : C) {Y Y' : Cᵒᵖ} (f : Y ⟶ Y') :
  (yoneda.obj X).map f = λ g, f.unop ≫ g := rfl
@[simp] lemma map_app {X X' : C} (f : X ⟶ X') (Y : Cᵒᵖ) :
  (yoneda.map f).app Y = λ g, g ≫ f := rfl
```

These three lemmas can be automatically generated by the `@[simps]` attribute.

Tools: Semantic Linters

We have a suite of semantic linters: they look through mathlib for common mistakes.

They run on every new commit to mathlib.

Some mistakes that it catches:

- A lemma has a hypothesis that is never used;
- A definition is incorrectly marked as a lemma;
- A definition has no documentation string;
- You created a loop in the simplification lemmas;
- You created a loop in the type-class search.
- ...

Design Decisions

In mathlib we have made some design decisions to make it convenient to formalize mathematics.

- Most of the library uses classical logic.
- Definitional equality is used sparingly.
- Dependent types and quotients are used to define general types. Example: $L^1(X, Y; \mu)$.
- We have a single repository where all components can work together.

Refactoring mathematics

One important feature of mathlib is to reduce duplication of proofs.

- Definitions and proofs should be in the greatest generality possible (within reason).
- This regularly requires refactoring of mathematics.
- There are also regular large-scale refactors on the library to make basic definitions more convenient or more general.

Refactoring mathematics: limits

$$\lim_{x \to x_0} f(x) = y_0 \qquad \lim_{x \to x_0^-} f(x) = -\infty$$

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = y_0 \qquad \lim_{x \to -\infty} f(x) = y_0^+$$

There are many different versions of limits.

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There are many different versions of limits.

These can all be unified by defining limits in terms of filters.

$$\begin{aligned} \mathcal{N}_x &= \text{neighborhoods of } x \\ \mathcal{N}_x^+ &= \{A \mid A \cap [x, \infty) \text{ is a neighborhood of } x \text{ in } [x, \infty)\} \\ \mathcal{N}_{+\infty} &= \{A \mid \exists y,\ [y, \infty) \subseteq A\} \end{aligned}$$

$f : X \to Y$ converges to a filter G on Y along filter F on X if

$$\forall A \in G,\ f^{-1}(A) \in F.$$

Type Classes

A major component of reusability of definitions is the use of type-classes.

This allows us to define concepts and prove theorems about general types with some structure.

Applications:

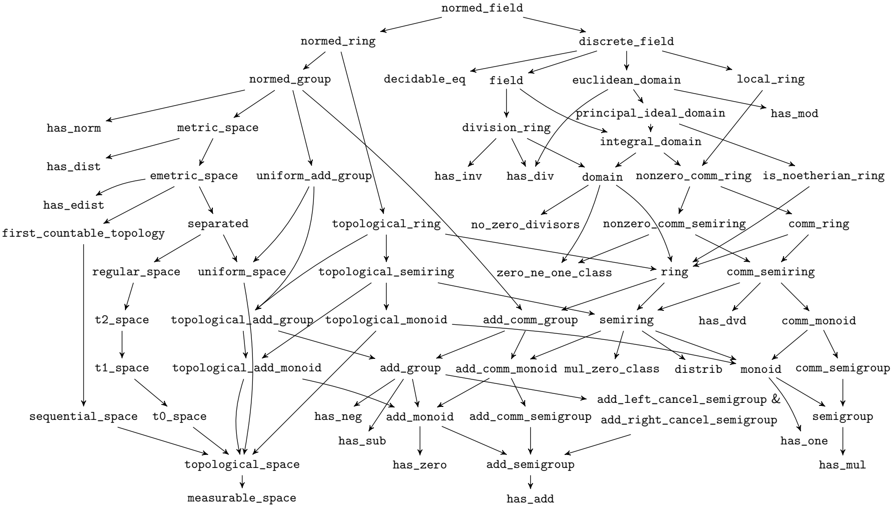
- Common operations: like `a * b`, $\|$`x`$\|$, $\bigsqcup$ `i, A i`.
- General theory: `group`, `normed_space`, `complete_lattice`
- Decidable propositions for implementing decision procedures.

They play a similar role as canonical structures in Coq.

Type Classes

```
class has_mul (A : Type) := (mul : A → A → A)

class semigroup (A : Type) extends has_mul A :=
(mul_assoc : ∀ a b c, (a * b) * c = a * (b * c))

class monoid (A : Type) extends semigroup A, has_one A :=
(one_mul : ∀ a, 1 * a = a) (mul_one : ∀ a, a * 1 = a)

variables {A : Type} [monoid A]
def pow (a : A) : ℕ → A
| 0     := 1
| (n+1) := a * pow n

theorem pow_add (a : A) (m : ℕ) : ∀ n, a ^ (m + n) = a ^ m * a ^ n
| 0     := by rw [add_zero, pow_zero, mul_one]
| (n+1) := by rw [add_succ, pow_succ, pow_add, mul.assoc]

instance : linear_ordered_comm_ring ℤ := ...
```

Graph



A part of the type-class hierarchy in October 2019.

Type Classes

In total there are 760 classes with more than 11000 instances between the classes.

They are used essentially everywhere.

Classes typically have a type as argument, like `topological_space X`.

There are also many "mixin" type-classes that depend on another type class, like `second_countable_topology X`, `compact_space X`, `t2_space X`, `connected_space X`, ...

In fact, there are 25 classes that depend on `topological_space` (and no other classes).

There are also many classes that depend on multiple other classes:
`[ring R] [topological_space R] [topological_ring R]`

Example: intermediate fields

In algebra, you often work with field extensions. L/K (L is an extension of K) means that K is a subfield of L. Example: $\mathbb{C}/\mathbb{R}$.

One often has to work with multiple extensions: F is an intermediate field if L/F and F/K.

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One option: F and K are subsets of the type L.
Problem: inconvenient in practice: the type $\mathbb{R}$ is not the same as the subset $\mathbb{R} \subseteq \mathbb{C}$.

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```
variables {K F L : Type*}
variables [field K] [field F] [field L]
variables [algebra K F] [algebra F L] [algebra K L]
variables [is_scalar_tower K F L]
```

Lean 4

A new version of Lean was released this year.

It is a full-fledged dependently-typed programming language, with a compiler to C.

It is highly extensible, with a flexible macro system.

It is not backwards compatible, but mathlib will be ported to Lean 4 next year:

- We can already port compiled Lean 3 files to compiled Lean 4 files (binport).
- There is an experimental method to port source Lean 3 to source Lean 4 files (synport).
- We need to manually implement the Lean 3 tactics in Lean 4: the meta-language has changed significantly.

Thank You