The Sobolev inequality in Lean

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Subtitle: In search of Fubini's theorem for finitary products.

Overview:

- Some measure theory preliminaries
- Products of measures
- The Gagliardo-Nirenberg-Sobolev inequality
- The marginal construction

Background: Measure Theory

Definition

A σ-algebra Σ on X is a collection of subsets of X that contains the empty set and is closed under complements and countable unions.
In Lean, writing `[MeasurableSpace X]` equips `X` with a σ-algebra of measurable sets.

Definition

If Σ is a σ-algebra on X, then a measure on Σ is a function $\mu : \Sigma \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and μ is *countably additive*: For pairwise disjoint sets $\{A_i\}_i$ we have

$$\mu \left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

Lebesque integral

Definition (Lebesque integral)

If $g : X \to [0, \infty]$ is a *simple function* (a function with finite range whose level sets are measurable), we can define

$$\int g \, d\mu = \int g(x) \, \mu(dx) = \sum_{y \in g(X)} \mu(g^{-1}\{y\}) \cdot y \in [0, \infty].$$

If $f : X \to [0, \infty]$ is any function, we can define the (lower) Lebesgue integral of f as the supremum of $\int g \, \mu(dx)$ for all simple $g \leq f$ (pointwise).

Bochner Integral

Definition (Integrable)

If E is a Banach space then we call a function $f : X \to E$ μ-integrable if f is the pointwise limit of simple functions and $\int \|f\| \, d\mu < \infty$.

Definition (Bochner integral)

For integrable functions we can define the Bochner integral $\int f \, d\mu \in E$ in a similar way to the Lebesgue integral.

Product measures

If μ is a measure on X and ν a measure on Y then we can define the product measure $\mu \times \nu$ on $X \times Y$. It can be defined as

$$(\mu \times \nu)(C) = \int_X \nu\{y \mid (x, y) \in C\}\, \mu(dx).$$

For general measures there are multiple product measures, but if μ and ν are σ-finite, then there is a unique product measure satisfying

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B).$$

Tonelli's theorem

Theorem (Tonelli's theorem)

Let $f : X \times Y \to [0, \infty]$ *be a measurable function.*
Then

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_X \int_Y f(x,y)\,\nu(dy)\mu(dx) = \int_Y \int_X f(x,y)\,\mu(dx)\nu(dy),$$

and all the functions in the integrals above are measurable.

Fubini's theorem

Theorem (Fubini's theorem)

Let E be a Banach space and $f : X \times Y \to E$ be an integrable function. Then

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_X \int_Y f(x,y)\, \nu(dy)\mu(dx) = \int_Y \int_X f(x,y)\, \mu(dx)\nu(dy),$$

Moreover, all the functions in the integrals above are measurable.

Remark. $f : X \times Y \to E$ is integrable iff the following two conditions hold:

- for almost all $x \in X$ the function $y \mapsto f(x,y)$ is integrable;
- The function $x \mapsto \int_Y \|f(x,y)\|\, \nu(dy)$ is integrable.

Iterated products

Suppose we have finitely many measure spaces $(X_i, \mu_i)_{i \in I}$ and we want to define a measure on $\prod_i X_i$. We could choose an ordering $I = \{i_1, \ldots, i_n\}$ and define the measure as (roughly) $\mu_{i_1} \times \cdots \times \mu_{i_n}$, but that means we have a non-canonical choice in the definition.

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We take the maximal measure μ such that for all $A_i \subseteq X_i$ we have

$$\mu(\Pi_i A_i) \leq \prod_i \mu_i(A_i).$$

We then show that equality holds by using the above non-canonical measure.

Iterated products

How would we state Tonelli's and Fubini's theorem for iterated products? We want something like

$$\int_{\mathbb{R}^{n+m}} f(z)\,dz = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x,y)\,dydx.$$

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The canonical equivalence $e : \mathbb{R}^{n+m} \simeq \mathbb{R}^n \times \mathbb{R}^m$ preserves the Lebesgue measure, so

$$\begin{aligned}\int_{\mathbb{R}^{n+m}} f(z)\,dz &= \int_{\mathbb{R}^n \times \mathbb{R}^m} f(e^{-1}(z))\,dz \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x,y)\,dydx.\end{aligned}$$

Iterated products

However, we want something more general:

- We should be able to pull out any k of the components and integrate over those variables, and then integrate over everything else. Even stating that precisely is not easy!
- We also want to generalize to finite products of Banach spaces, not just $\mathbb{R}^n$

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It was unclear how to formulate this, so we didn't have any version of Fubini's theorem for iterated products.

Gagliardo-Nirenberg-Sobolev inequality

The L^p-norm of a function f is

$$\|f\|_{L^p} := \left(\int \|f\|^p\right)^{\frac{1}{p}} \in [0, \infty]$$

Theorem (Gagliardo-Nirenberg-Sobolev inequality)

Let E be a real normed space of finite dimension $n \geq 2$. Let $1 \leq p < n$ be a real number and $p^* = \frac{np}{n-p}$ its Sobolev conjugate. Then there exists a nonnegative real number C such that for all compactly supported C^1 functions $u : E \to \mathbb{R}$, we have

$$\|u\|_{L^{p^*}} \leq C\|Du\|_{L^p}$$

Application: Sobolev spaces

Let $u : \Omega \to \mathbb{R}$ for some nice open $\Omega \subseteq \mathbb{R}^n$.
If u is differentiable with derivative $v = Du : \Omega \to \mathbb{R}^n$, then for all compactly supported smooth functions $\varphi : \Omega \to \mathbb{R}$

$$\int_\Omega v\varphi = -\int_\Omega uD\varphi.$$

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The Sobolev space $H_0^1(\Omega, \mathbb{R})$ consists of all functions that have an L^2 weak derivative and are a L^2 limit of smooth functions with compact support.

Application: Sobolev spaces

The Sobolev inequality implies that for $u \in H_0^1(\Omega)$

$$\|u\|_{L^2} \leq \|u\|_{L^{\frac{2n}{n-2}}} \leq C\|Du\|_{L^2}.$$

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The Sobolev space is a Hilbert space with

$$\|u\|_{H_0^1} := \sqrt{\|u\|_{L^2}^2 + \|Du\|_{L^2}^2} \le C'\|Du\|_{L^2}.$$

This means that $(u, v) \mapsto \langle Du, Dv\rangle_{L^2}$ forms an inner product on H_0^1 that gives the same topology.

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For $f \in L^2(\Omega)$ we have that the operator $v \mapsto \langle f, v\rangle_{L^2} : H_0^1(\Omega) \to \mathbb{R}$ is a bounded linear functional on $H_0^1(\Omega)$. By the Riesz representation theorem there is a unique element $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$ we have

$$\langle Du, Dv\rangle_{L^2} = \langle f, v\rangle_{L^2}.$$

Partial Differential Equations

A fundamental elliptic PDE is the Poisson equation:

$$-\Delta u = f$$

where $f : \Omega \to \mathbb{R}$ and $\Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2}$.

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$$\int_\Omega \langle Du, D\varphi \rangle = \int_\Omega (-\Delta u)\varphi = \int_\Omega f\varphi.$$

We say that $u \in H_0^1(\Omega)$ is a weak solution to the PDE if for all such $v \in H_0^1(\Omega)$

$$\int_\Omega \langle Du, Dv \rangle = \int_\Omega fv.$$

Proof of the inequality

We prove the Gagliardo-Nirenberg-Sobolev inequality first for $p = 1$ and then use that in the general case.

For $p = 1$ one estimates

$$|u(x)| = \left|\int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x) dx_i\right| \leq \int_{\mathbb{R}} |Du(x)| dx_i.$$

Then one inductively integrates over the variables x_1, x_2, x_3, ... and applies Hölder's inequality multiple times.

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The induction hypothesis involves expressions like

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |Du(x)| dx_1 dx_2 \cdots dx_k dx_i$$

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If $f : \mathbb{R}^n \to \mathbb{R}$ we want to integrate it over some subset of the variables of $\{x_1, \ldots, x_n\}$.

Marginal construction: Definition

Let I be a indexing set, $A \subseteq I$ a finite subset and E be a Banach space. For $i \in I$ suppose we are given a measure space (X_i, μ_i). If $x \in \prod_{i \in I} X_i$ and $y \in \prod_{i \in A} X_i$ we write $x[y/A]$ for the vector

$$x[y/A]_i := \begin{cases} y_i & \text{if } i \in A \\ x_i & \text{otherwise.} \end{cases}$$

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Let $f : (\prod_{i \in I} X_i) \to [0, \infty]$ be a function. Then the *marginal of f w.r.t. A*

$$\int \cdots \int_{i \in A} f \, d\mu_i$$

is by definition another function $(\prod_{i \in I} X_i) \to [0, \infty]$ that is defined as

$$x \mapsto \int_{\prod_{i \in A} X_i} f(x[y/A]) \, d\Pi_{i \in A} \mu_i(y).$$

Marginal construction: Remarks

- We call this the marginal construction in reference to probability theory. If all the μ_i are probability measures and f is a random variable, then $\int\cdots\int_{i\in A} f\,\mathrm{d}\mu_i$ is the marginal variable on $\prod_{i\in I\setminus A} X_i$.
- Note that

$$\int\cdots\int_{i\in A} f\, d\mu_i : (\prod_{i\in I} X_i) \to [0,\infty],$$

  but it only depends on the arguments in X_i for $i \notin A$.
- The definition is ugly.
- It has very nice properties.

Marginal construction: Properties

Proposition

Let $f : (\prod_{i \in I} X_i) \to [0, \infty]$.

1. *If* $x, x' \in \prod_{i \in I} X_i$ *and* $x_i = x'_i$ *for all* $i \in I \setminus A$ *then* $\int \cdots \int_{i \in A} f \, \mathrm{d}\mu_i$ *will have the same value on* x *and* x'.

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4. *If* $i_0 \in I$ *then*

$$\int \cdots \int_{i \in \{i_0\}} f \, \mathrm{d}\mu_i = \int_{X_{i_0}} f(x[y/i_0]) \mathrm{d}\mu_{i_0}(y)$$

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$$\int\cdots\int_{i \in \{i_0\}} f \, \mathrm{d}\mu_i = \int_{X_{i_0}} f(x[y/i_0]) \mathrm{d}\mu_{i_0}(y)$$

5. *If* f *is measurable and* A *and* B *are disjoint finite subsets of* I, *then*

$$\int\cdots\int_{i \in A \cup B} f \, \mathrm{d}\mu_i = \int\cdots\int_{i \in A} \int\cdots\int_{j \in B} f \, \mathrm{d}\mu_j \, \mathrm{d}\mu_i$$

Marginal Construction: applications

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- The Gagliardo-Nirenberg-Sobolev inequality
- Compute the volume of the unit ball in $\mathbb{R}^n$ (this was also done independently by Xavier Roblot).
- We can shorten the proof of a lemma in the proof of the change of variables theorem that used Fubini manually in $\mathbb{R}^n$.
- We found a lemma tagged "Fubini's theorem" in HOL Light. It states that to compute a $\mu(s)$ for $s \subseteq \mathbb{R}^{n+1}$ we can take the measure of the set for a fixed i-th coordinate x and then integrate over x. In Lean:

```
theorem lintegral_measure_insertNth
  {s : Set (∀ i : Fin (n+1), α i)}
  (hs : MeasurableSet s) (i : Fin (n+1)) :
  ∫⁻ x, Measure.pi (μ ∘' i.succAbove)
    (insertNth i x ⁻¹' s) ∂μ i = Measure.pi μ s
```

(35 lines in Lean, 300-600 in HOL Light, the HOL Light version assumes that s is bounded).

Sample proof

```
calc ∫⁻ x : α i, Measure.pi (μ ∘' succAbove i) (insertNth i x ⁻¹' s) ∂μ i
    = ∫⁻ x : α i, (∫⋯∫⁻_.univ, indicator (insertNth i x ⁻¹' s) 1 ∂μ ∘' succAbove i) y ∂μ i := by
      simp_rw [← lintegral_indicator_one (measurable_insertNth _ hs),
        lintegral_eq_lmarginal_univ y]
  _ = ∫⁻ x : α i, (∫⋯∫⁻_.univ, indicator (insertNth i x ⁻¹' s) 1 ∂μ ∘' succAbove i)
        (z ∘' i.succAbove) ∂μ i := by
      rw [← insertNth_dcomp_succAbove i x y]
  _ = ∫⁻ x : α i, (∫⋯∫⁻_{i}ᶜ,
        indicator (insertNth i x ⁻¹' s) 1 ∘ (· ∘' succAbove i) ∂μ) z ∂μ i := by
      simp_rw [← λ x ↦ lmarginal_image succAbove_right_injective (μ := μ) .univ
        (f := indicator (insertNth i x ⁻¹' s) (1 : ((j : Fin n) → α (succAbove i j)) → ℝ≥0∞))
        (measurable_one.indicator (measurable_insertNth _ hs)) z, Fin.image_succAbove_univ]
  _ = ∫⁻ x : α i, (∫⋯∫⁻_{i}ᶜ,
        indicator (insertNth i x ∘ (· ∘' succAbove i) ⁻¹' s) 1 ∂μ) z ∂μ i := by
      rfl
  _ = ∫⁻ x : α i, (∫⋯∫⁻_{i}ᶜ,
        indicator ((Function.update · i x) ⁻¹' s) 1 ∂μ) z ∂μ i := by
      simp [comp]
  _ = (∫⋯∫⁻_insert i {i}ᶜ, indicator s 1 ∂μ) z := by
      simp_rw [lmarginal_insert _ (measurable_one.indicator hs) hi,
        lmarginal_update_of_not_mem (measurable_one.indicator hs) hi]
      rfl
  _ = (∫⋯∫⁻_.univ, indicator s 1 ∂μ) z := by simp
  _ = Measure.pi μ s := by rw [← lintegral_indicator_one hs, lintegral_eq_lmarginal_univ z]
```

Side conditions

```
let n' := NNReal.conjExponent n
have h0n : 2 ≤ n := Nat.succ_le_of_lt <| Nat.one_lt_cast.mp <| hp.trans_lt h2p
have hn : NNReal.IsConjExponent n n' := .conjExponent (by norm_cast)
have h1n : 1 ≤ (n : ℝ≥0) := hn.one_le
have h2n : (0 : ℝ) < n - 1 := by simp_rw [sub_pos]; exact hn.coe.one_lt
have hnp : (0 : ℝ) < n - p := by simp_rw [sub_pos]; exact h2p
rcases hp.eq_or_lt with rfl|hp
let q := Real.conjExponent p
have hq : Real.IsConjExponent p q := .conjExponent hp
have h0p : p ≠ 0 := zero_lt_one.trans hp |>.ne'
have h1p : (p : ℝ) ≠ 1 := hq.one_lt.ne'
have h3p : (p : ℝ) - 1 ≠ 0 := sub_ne_zero_of_ne h1p
have h0p' : p' ≠ 0 := by
  suffices 0 < (p' : ℝ) from (show 0 < p' from this) |>.ne'
  rw [← inv_pos, hp', sub_pos]
  exact inv_lt_inv_of_lt hq.pos h2p
have h2q : 1 / n' - 1 / q = 1 / p' := by
  simp_rw (config := {zeta := false}) [one_div, hp']
  rw [hq.conj_inv_eq, hn.coe.conj_inv_eq, sub_sub_sub_cancel_left]
  simp
let γ : ℝ≥0 := ⟨p * (n - 1) / (n - p), by positivity⟩
have h0γ : (γ : ℝ) = p * (n - 1) / (n - p) := rfl
have h1γ : 1 < (γ : ℝ) := by
  rwa [h0γ, one_lt_div hnp, mul_sub, mul_one, sub_lt_sub_iff_right, lt_mul_iff_one_lt_left]
  exact hn.coe.pos
have h2γ : γ * n' = p' := by
  rw [← NNReal.coe_inj, ← inv_inj, hp', NNReal.coe_mul, h0γ, hn.coe.conj_eq]
  field_simp; ring
```

Final thoughts

- The marginal construction is a very convenient tool to deal with iterated integrals.
- It allows one to conveniently apply Fubini's theorem
- We want to properly formalize Sobolev spaces and their applications to PDEs.

Thank You