Formalized Spectral Sequences in Homotopy Type Theory

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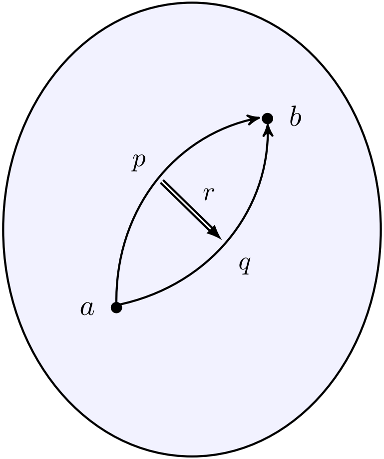
Joint work with Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke and Mike Shulman.

Recap: Path spaces

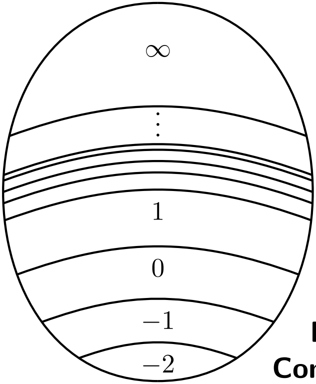
A type A can have

- points $a, b : A$
- paths $p, q : a = b$
- paths between paths $r : p = q$

$$\vdots$$



Recap: Truncated Types



$(n+1)$-**Type**: all paths n-types

1-Type: all paths are sets

Set: satisfies UIP / axiom K

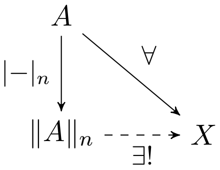
Proposition: as at most one point

Contractible: has exactly one point

Recap: Truncation

Given A, we can form the n-truncation $\|A\|_n$.

$\|A\|_n$ is the "best approximation" of A which is n-truncated.

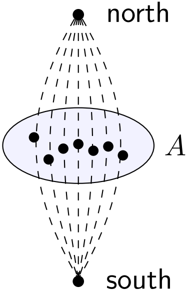


Recap: The suspension

We have Higher inductive types (HITs), like the suspension ΣA.

$\texttt{HIT}\ \Sigma A :\equiv$

- $\mathsf{north}, \mathsf{south} : \Sigma A$
- $\mathsf{merid} : A \to (\mathsf{north} = \mathsf{south})$



Recap: Pointed types and maps

Definition If $f : X \to Y$ and $y : Y$, the fiber of f at y is $\mathsf{fib}_f(y) :\equiv \Sigma(x : X),\ f(x) = y$.

Definition An element of $\Sigma(X : \mathrm{Type}),\ X$ is called a pointed type.

Definition If X is a pointed type, its loop space is $\Omega X :\equiv (x_0 = x_0, \mathsf{refl}_{x_0})$.

Definition If X and Y are pointed types, a the type of pointed maps $X \to^* Y$ is defined as $\Sigma(f : X \to Y),\ f(x_0) = y_0$.

Cohomology

How do we define (co)homology?

The usual constructions are not homotopy invariant.

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Theorem. The cohomology groups $H^n(X;G)$ are naturally equivalent to homotopy classes of maps $[X, K(G,n)]$.

$K(G,n)$ is the an *Eilenberg-Maclance space*, which is the (unique up to homotopy equivalence) space X with $\pi_n(X) = G$ and $\pi_k(X) = 0$ for $k \neq n$.

Eilenberg-MacLane spaces are usually defined as CW-complexes.

Example. $K(\mathbb{Z},1) = \mathbb{S}^1$.

Eilenberg-MacLane spaces

We can define $K(G, n)$ in HoTT. We first define the following higher inductive type:

HIT $\widetilde{K}(G, 1) :\equiv$

- $\star : \widetilde{K}(G, 1)$
- pth : $G \to (\star = \star)$
- pth-mul : $\Pi(g\ h : G),\ \mathsf{pth}(gh) = \mathsf{pth}(g) \cdot \mathsf{pth}(h)$

Then $K(G, 1) :\equiv \|\widetilde{K}(G, 1)\|_1$.

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For $n \geq 1$ we can define $K(G, n+1) :\equiv \|\Sigma K(G, n)\|_{n+1}$ (if G is abelian).

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For $n \geq 1$ we can define $K(G,n+1) :\equiv \|\Sigma K(G,n)\|_{n+1}$ (if G is abelian).

Theorem. $K(G,n)$ is the unique n-truncated pointed type X with $\pi_n(X) = G$ and $\pi_k(X) = 0$ for $k \neq n$.
A useful property: $K(G,n) = \Omega K(G,n+1)$, which gives a "multiplication" on $K(G,n)$

Cohomology

We can now define the reduced cohomology of a pointed type X with coefficients in an abelian group G to be

$$\widetilde{H}^n(X,G) :\equiv \|X \to^* K(G,n)\|_0.$$

The unreduced cohomology can be defined similarly for any (not necessarily pointed) type X:

$$H^n(X,G) :\equiv \|X \to K(G,n)\|_0 = \widetilde{H}^n(X+1,G).$$

The group structure comes from $K(G,n)$.

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Remark. We can also define reduced homology:

$$\widetilde{H}_n(X,G) :\equiv \mathsf{colim}_k\big(\pi_{n+k}(X \wedge K(G,n+k))\big).$$

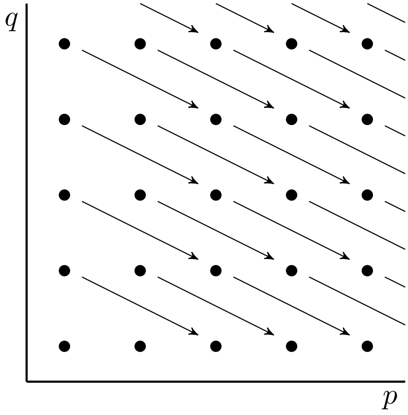
Here $\wedge$ is the smash product.

Spectral Sequences

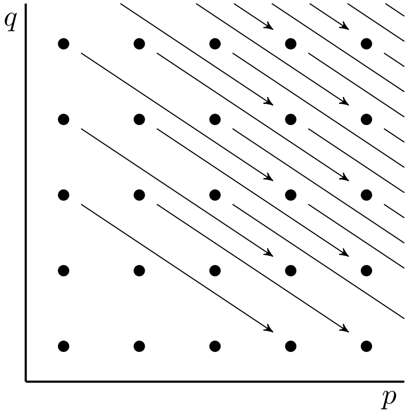
Definition A (cohomologically indexed) spectral sequence consists of

- A family $E_r^{p,q}$ of abelian groups (or more generally: R-modules) for $p, q : \mathbb{Z}$ and $r \geq 2.$ For a fixed r this gives the r-*page* of the spectral sequence.
- *differentials* $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ with $d_r \circ d_r = 0.$
- isomorphisms $\alpha_r^{p,q} : H^{p,q}(E_r) \simeq E_{r+1}^{p,q}$ where $H^{p,q}(E_r) = \mathsf{ker}(d_r^{p,q})/\mathsf{im}(d_r^{p-r,q+r-1}).$

Spectral Sequences



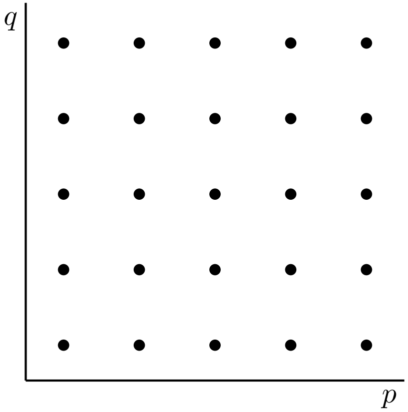
$E_2^{p,q}$



$E_3^{p,q}$

Convergence of Spectral Sequences

The pages converge to $E_\infty^{p,q}$.



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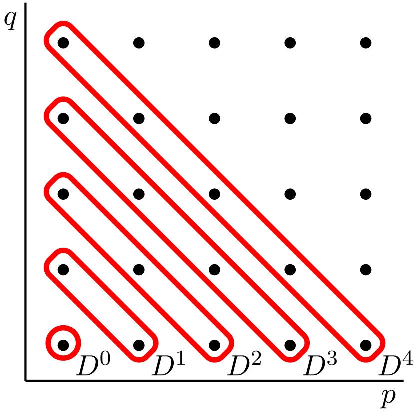
We can get information about the diagonals on the infinity page.



$E_\infty^{p,q}$

Convergence of Spectral Sequences

For a bigraded abelian group $C^{p,q}$ and graded abelian group D^n we write

$$E_2^{p,q} = C^{p,q} \Rightarrow D^{p+q}$$

if there exists a spectral sequence $E_r^{p,q}$ such that

- The second page is $E_2^{p,q} = C^{p,q}$
- D^n is built up from $E_\infty^{p,q}$ for $n = p + q$ in the following way:

  We have short exact sequences:

$$\begin{aligned} E_\infty^{0,n} &\to D^n \to D^{n,1} \\ &\vdots \\ E_\infty^{p,q} &\to D^{n,p} \to D^{n,p+1} \\ E_\infty^{p+1,q-1} &\to D^{n,p+1} \to D^{n,p+2} \\ &\vdots \\ E_\infty^{n,0} &\to D^{n,n} \to 0 \end{aligned}$$

Serre Spectral Sequence (special case)

Theorem. Suppose $f : X \to B$ and $b_0 : B$.
Let $F :\equiv \mathsf{fib}_f(b_0) :\equiv \Sigma(x : X),\ f(x) = b_0$ be the fiber of f at b_0.
Suppose that B is *simply connected*, i.e. $\|B\|_1$ is contractible. Then

$$E_2^{p,q} = H^p(B, H^q(F, G)) \Rightarrow H^{p+q}(X, G).$$

This is the *unreduced* cohomology.

Example: cohomology of $K(\mathbb{Z}, 2)$

We will compute the cohomology groups of $B = K(\mathbb{Z}, 2)$ (which is $\mathbf{CP}^\infty$). We define the map $1 \xrightarrow{f} K(\mathbb{Z}, 2)$ determined by the basepoint $b_0 : K(\mathbb{Z}, 2)$. It has fiber

$$\begin{aligned} &\big(\Sigma(x:1),\ f(x) = b_0\big) \\ &= (f(\star) = b_0) \\ &= \Omega K(\mathbb{Z}, 2) \\ &= K(\mathbb{Z}, 1) \\ &= \mathbb{S}^1. \end{aligned}$$

The spectral sequence for $G = \mathbb{Z}$ gives

$$E_2^{p,q} = H^p(B, H^q(\mathbb{S}^1)) \Rightarrow H^{p+q}(1).$$

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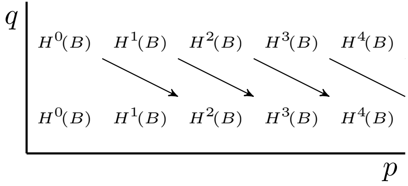

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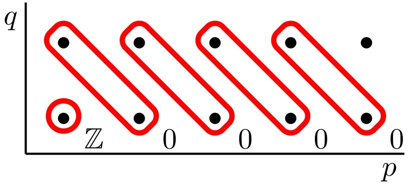

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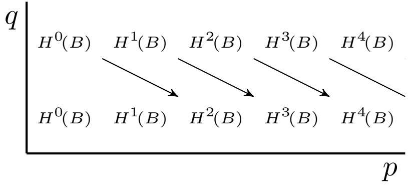

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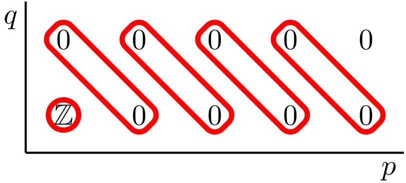

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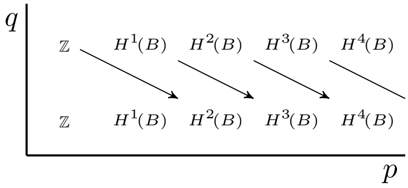

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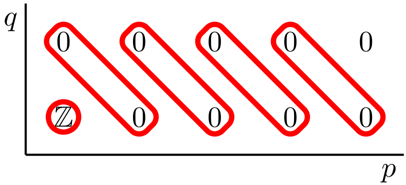

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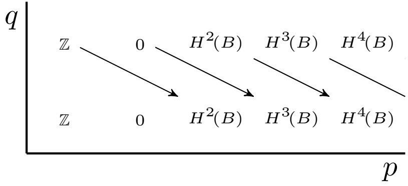

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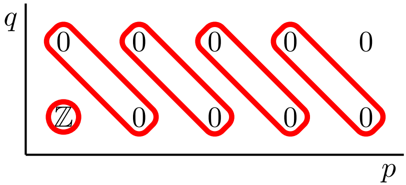

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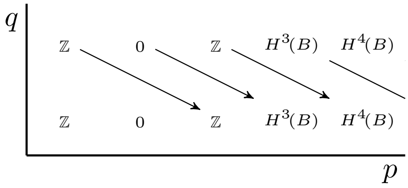

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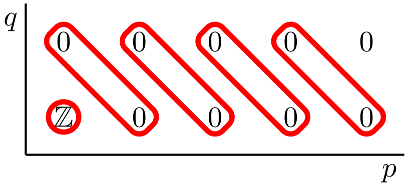

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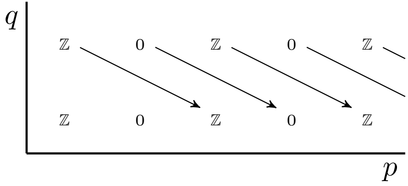

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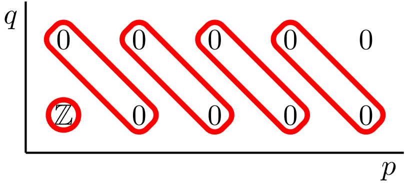

$E_\infty^{p,q}$

Spectra

For the general Serre spectral sequence, we need to generalize cohomology.

We need generalized and parametrized cohomology.

An (omega)-spectrum is a sequence of pointed types $Y : \mathbb{N} \to \mathrm{Type}^*$ such that $\Omega Y_{n+1} = Y_n$.

Example. $Y_n = K(G, n)$ is a spectrum.

A spectrum is called n-truncated if Y_k is $(n + k)$-truncated for all $k : \mathbb{N}$.

Now suppose X is a type and $Y : X \to \mathrm{Spectrum}$ is a *family of spectra* over X.

We can define $H^n(X, \lambda x.Y x) :\equiv \|\Pi(x : X),\ Y_n(x)\|_0$.

Theorem. *(Serre Spectral Sequence)* If $f : X \to B$ is any map and Y is a truncated spectrum, then

$$E_2^{p,q} = H^p(B, \lambda b. H^q(\mathsf{fib}_f(b), Y)) \Rightarrow H^{p+q}(X, Y).$$

If $Y_n = K(G, n)$ and B is simply connected and pointed, then this reduces to the previous case

$$E_2^{p,q} = H^p(B, H^q(\mathsf{fib}_f(b_0), G)) \Rightarrow H^{p+q}(X, G).$$

Atiyah-Hirzebruch Spectral Sequence

For a spectrum Y, its homotopy groups are $\pi_n(Y) :\equiv \pi_{n+k}(Y_k)$ (which is independent of k and also defined for negative n).

Special case. If X is any type and Y is a truncated spectrum, then

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Theorem. *(Atiyah-Hirzebruch Spectral Sequence)* If X is any type and $Y : X \to \mathrm{Spectrum}$ is a family of truncated spectra over X, then

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The Atiyah-Hirzebruch spectral sequence is also true if we replace all cohomologies by reduced cohomologies.

HoTT in proof assistants

There are various proof assistants supporting HoTT

- Coq (UniMath and Coq-HoTT)
- Agda
- Lean
- cubicaltt
- RedPRL

The Lean Theorem Prover

Lean is a new interactive theorem prover, developed principally by Leonardo de Moura at Microsoft Research.

It was "announced" in the summer of 2015.

It is open source, released under a permissive license, Apache 2.0.

We have formalized the HoTT library in a previous version of Lean, "Lean 2".
We are currently working in porting it to the newest version, "Lean 3".

The Lean Theorem Prover

Notable features:

- implements dependent type theory
- written in C++, with multi-core support
- small, trusted kernel and multiple independent type checkers
- powerful elaborator
- can use proof terms or tactics
- editors with proof-checking on the fly
- browser version runs in javascript
- use Lean as a programming language to write programs, for example tactics and automation for proofs

The HoTT library

The HoTT library ($\sim$47k LOC) contains

- A good library with the basics of homotopy type theory
- A category theory library
- A large library for synthetic homotopy theory. Sample:
  - Freudenthal suspension theorem
  - Whitehead's theorem
  - Seifert-van Kampen theorem
  - $\pi_k(\mathbb{S}^n)$ for $k \leq n$ and $\pi_3(\mathbb{S}^2)$.
  - adjunction between the smash product and pointed maps.
  - the Serre spectral sequence

Contributors: vD, Jakob von Raumer, Ulrik Buchholtz, Jeremy Avigad, Egbert Rijke, Steve Awodey, Mike Shulman and others.

Formalization

- We started the formalization of the Serre spectral sequence almost 2 years ago, in November 2015.
- vD, Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke and Mike Shulman have actively worked on the formalization.
- Most time was spent on basic results like group theory, graded R-modules, and basic properties of spectra and types.
- It is not clear how long the formalization is: many results can be reused elsewhere.

Future work

- Provide a good "interface" for spectral sequences;
- Port the result to the current version of Lean;
- The cup product structure on cohomology;
- Homological Serre spectral sequence;
- Applications of the Serre spectral sequence:
  - Serre class theorem
  - Hurewicz theorem
  - computation of $\pi_{n+k}(\mathbb{S}^n)$ for $k \leq 3$.

Thank you