Constructing the Propositional Truncation using Non-recursive HITs

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Abstract

In homotopy type theory, we construct the propositional truncation as a colimit, using only non-recursive higher inductive types (HITs). This is a first step towards reducing recursive HITs to non-recursive HITs. This construction gives a characterization of functions from the propositional truncation to an arbitrary type, extending the universal property of the propositional truncation. We have fully formalized all the results in a new proof assistant, Lean.

Categories and Subject Descriptors F.4.1 [*Mathematical Logic*]

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1. Introduction

Homotopy Type Theory (HoTT) is based on a connection of intensional type theory with homotopy theory and higher category theory [AW09]. In HoTT a type can be viewed as a topological space, up to homotopy. In this setting the notion of an inductive type can be generalized to a notion of a *Higher Inductive Type* (HIT) [Uni13]. When defining a HIT, you can specify not only the point constructors, but also the path constructors of a type. For example, you can define the circle S^1 as a HIT with one point constructor and one path constructor, generated by:

- base : S^1
- loop : base $=$ base

Using the Univalence Axiom [Voe14], you can prove that loop $\neq \text{refl}_{\text{base}}$, which means the circle is not just the unit type.

Another HIT is the propositional truncation $\|A\|$ of a type A. An inhabitant of $\|A\|$ indicates that A is inhabited, without specifying a particular inhabitant. The type $\|A\|$ is always a *mere proposition*, meaning that any two inhabitants are equal. The propositional truncation is similar to the bracket type [AB04] in extensional type theory and to the squash type [CAB+86] in NuPRL. The propositional truncation of a type A can be specified as a HIT with these constructors:

- $|-| : A \to \|A\|$
- $\varepsilon : \Pi(x, y : \|A\|),\ x = y$

Note that the path constructor quantifies over all elements of the type $\|A\|$, that is, the type we are defining. This means that it is a *recursive* HIT because we can apply the path constructor ε recursively. An example of recursive application is (loosely speaking)

$$\text{apd}_{\varepsilon(x)}(\varepsilon(y,z)) : \varepsilon(x,y) \cdot \varepsilon(y,z) = \varepsilon(x,z) \tag{1}$$

for $x, y, z : \|A\|$.

HITs are not very well understood. There is no general theory of HITs that specifies which HITs are allowed, or what form the constructors can have. Giving such a theory would also require giving a model where all such HITs exist to ensure consistency of the resulting type theory. To do it, it might be useful to use a reductive approach. Suppose we can reduce a broad class of HITs to a few particular HITs. In this case, we only need a model of these particular HITs to get a model of the broader class of HITs. This is similar to the situation for inductive types in extensional type theory. Every inductive type in extensional type theory can be reduced to Σ-types and W-types [Dyb97], so a model which has Σ-types and W-types has all inductive types.

This paper is a first step in such a reductive approach. In this paper we reduce the propositional truncation — the prototypical recursive HIT — to just two non-recursive HITs, the *sequential colimit* and the *one-step truncation*, which we will define now.

The one-step truncation $\{A\}$ of a type A is a non-recursive version of the propositional truncation. It has the following constructors:

- $f : A \to \{A\}$
- $e : \Pi(x, y : A),\ f(x) = f(y)$

The difference between the propositional truncation and the one-step truncation is that the path constructor of the former quantifies over all elements in the newly constructed type, while the path constructor of the latter only quantifies over all elements of A. This means that $\{A\}$ only adds a path between any two points already existing in A. In $\{A\}$ we do not add higher paths in the same way as in $\|A\|$, e.g. we cannot form an equality analogous to (1) in $\{A\}$.

We can easily give the universal property of $\{A\}$. We call a function $g : A \to B$ *weakly constant* if for all $a, a' : A$ we have $g(a) = g(a')$. The attribute *weakly* comes from the fact that we do not impose other conditions about these equality proofs (in Section 5 we discuss some other notions of constancy). Then the maps $\{A\} \to B$ correspond exactly to the weakly constant functions from A to B.

We can also define the (sequential) colimit as a HIT. Given a sequence of types $A : \mathbb{N} \to \mathcal{U}$ with maps $f : \Pi(n : \mathbb{N}),\ A_n \to A_{n+1}$ the colimit is the HIT with the following constructors:

- $i : \Pi(n : \mathbb{N}),\ A_n \to \text{colim}(A, f)$

- $g : \Pi(n : \mathbb{N}),\ \Pi(a : A_n),\ i_{n+1}(f_n(a)) = i_n(a)$

We will sometimes leave the arguments from $\mathbb{N}$ implicit for f, i and g.

Our construction of the propositional truncation is as follows. Given a type A, we can form the sequence

$$A \xrightarrow{f} \{A\} \xrightarrow{f} \{\{A\}\} \xrightarrow{f} \{\{\{A\}\}\} \xrightarrow{f} \cdots \tag{2}$$

Then the colimit $\{A\}_\infty$ of this sequence is the propositional truncation of A. What we mean by this is that the type $\{A\}_\infty$ satisfies exactly the formation, introduction, elimination and computation rules used to define the propositional truncation. From this we conclude:

- If we work in a type theory which does not have a propositional truncation operator, then we can define the map the propositional truncation to be the map $A \mapsto \{A\}_\infty$.
- If we already have a propositional truncation operator, then $\{A\}$ and $\|A\|$ are equivalent types.

We will give an intuition why this construction works in Section 3. The proof that the construction is correct (given in Section 4) uses function extensionality, which is a consequence of the Univalence Axiom [Uni13, Section 4.9]. Our construction does not otherwise use the Univalence Axiom (UA) itself, but some related results in this paper do, and in those case we will explicitly mention that we use UA.

This construction has multiple consequences. We already discussed this work as a starting point for a reductive approach to HITs. Another corollary is a generalization of the universal property of the propositional truncation. The universal property of the propositional truncation states that the type $\|A\| \to B$ is equivalent to $A \to B$ for mere propositions B. However, this leaves open the question what the type $\|A\| \to B$ is for types B which are not mere propositions. The construction in this paper answers this question: the functions in $\|A\| \to B$ are precisely the cocones over the sequence (2). A different answer to this question is given in [Kra15] (see Section 7 for a comparison). Another corollary of the construction is the following theorem. Given a weakly constant function $A \to A$, then A has split support, meaning $\|A\| \to A$. This is a known result [KECA14, Theorem 4.5], but we give an alternative proof in Section 5.

We have fully formalized this construction in the Lean proof assistant. Lean [dKA+15] is a interactive theorem prover under development by Leonardo de Moura at Microsoft Research. It is based on a version of the calculus of inductive constructions, like Coq and Agda. It has two libraries, a standard library for constructive and classical mathematics, and a HoTT library for homotopy type theory. In the HoTT library we are trying a new way to deal with HITs. Instead of Dan Licata's trick [Lic11] we are experimenting with the reductive approach. We have two primitive HITs: "quotients" (not to be confused with set-quotients) and the n-truncation. From this, we can define all other commonly used HITs. For more information about the formalization and Lean, see Section 6.

2. Preliminaries

In this section we will present some basic definitions from [Uni13] used in this paper. We follow the informal style of writing proofs from [Uni13]. In particular, if $f : A \to B \to C$ is a binary function, we write $f(x, y)$ for the application $(f\ x)\ y$.

The basis of HoTT is intensional Martin Löf Type Theory. The *path type* (also called the *identification type*, *identity type*, *equality type*) is an inductive type which for each type A and element $a : A$ gives a type family $\mathrm{Id}_A(a) : A \to \mathcal{U}$ which is generated by $\mathrm{refl}_a : \mathrm{Id}_A(a, a)$. The type $\mathrm{Id}_A\, ab$ is also written $a =_A b$ or $a = b$. Elements $p : a = b$ are called *paths*, *identifications* or *equalities*. Elements of $a = a$ are called *loops*. The identity type is *intensional*, which means that it can have multiple inhabitants which are not equal. The type $a = b$ should not be confused with the judgement $a \equiv b$ stating that a and b are *definitionally* or *judgmentally* equal, which is a meta-theoretic concept. We can also form the path type between to paths. That is, if $p, q : a =_A b$, then $p =_{a=_A b} q$ is a *2-dimensional path type*. In a similar manner we can define the *higher dimensional path types*. The following rule is the *path induction* principle for the identity type.

$$\frac{\begin{array}{ccc} A : \mathcal{U} & a : A & P : \Pi(b : A),\ a =_A b \to \mathcal{U} \\ \rho : P(a, \mathrm{refl}_a) & b : A & p : a =_A b \end{array}}{\mathrm{ind}_=(P, \rho, b, p) : P(b, p)}$$

Informally, path induction states that if we want to prove some property P about an arbitrary path, it is sufficient to prove it for reflexivity paths. The corresponding computation rule is $\mathrm{ind}_=(P, \rho, a, \mathrm{refl}_a) \equiv \rho$.

Using the path induction principle, we can define the following basic operations. Given elements $x, y, z : A$ and paths $p : x = y$ and $q : y = z$ we define the *concatenation* $p \cdot q : x = z$ and the *inverse* $p^{-1} : b = a$. If $f : A \to B$ is a function we can apply it to paths to get $\mathrm{ap}_f(p) : f(x) = f(y)$. If $P : A \to \mathcal{U}$ is given then p induces a function $P(x) \to P(y)$. In particular, if $u : P(x)$, then we have $\mathrm{transport}^P(p, u) : P(y)$. We also write $p_*(u)$ for $\mathrm{transport}^P(p, u)$. These definitions satisfy the obvious coherence laws such as $\mathrm{refl}_x \cdot p = p$ and $\mathrm{ap}_f(p^{-1}) = (\mathrm{ap}_f(p))^{-1}$.

For two functions $f, g : A \to B$ we write $f \sim g$ for the type of *homotopies* between f and g, which are the proofs that f and g are pointwise equal: $\Pi(x : A),\ f(x) = g(x)$. A function $f : A \to B$ is an *equivalence* if it has a left and a right inverse, i.e. if there are functions $g, h : B \to A$ such that $g \circ f \sim \mathrm{id}_A$ and $f \circ h \sim \mathrm{id}_B$. If f is an equivalence, then f also has a two-sided inverse, written f^{-1}. The fact that f is an equivalence is written $f : A \simeq B$, and A and B are called *equivalent* types.

If f and g are equal functions, then f and g are homotopic. This means that we have a map $(f = g) \to (f \sim g)$. *Function Extensionality* is the axiom that this map is an equivalence. Similarly, if two types A and B are equal, then they are equivalent, so we have a map $(A = B) \to (A \simeq B)$. The *Univalence Axiom* states that this map is an equivalence.

The types in a universe are stratified into a hierarchy of n-types for $n \geq -2$. A type A is a (-2)-type or *contractible* if it has a unique point, that is, if $\Sigma(x : A),\ \Pi(y : A),\ x = y$. In this case we can prove that $A \simeq \mathbf{1}$, where $\mathbf{1}$ is the unit type. A type A is an $(n+1)$-type if all its path types are n-types, i.e., if $\Pi(x\ y : A),\ \text{is-}n\text{-type}(x = y)$. This hierarchy is inclusive in the sense that every n-type is an $(n+1)$-type. The n-types are the types where the path structure becomes trivial for high dimensions, all paths above dimension $n+2$ are trivial (inhabited by a unique element, i.e. contractible).

The (-1)-types are called *mere propositions*, and a type is a mere proposition iff all its inhabitants are equal, i.e. $\Pi(x\ y : A),\ x = y$. The 0-types are called *sets*, and are the types where all equality types are mere propositions, i.e. where the uniqueness of identity proofs holds. A type A is a set iff it satisfies *Axiom K*: every loop is equal to reflexivity. The 1-types are the types where the equality types are sets, and so on.

Example 2.1.

- the natural number $\mathbb{N}$ and the booleans $\mathbf{2}$ are sets, but not mere propositions.
- The circle S^1 is a 1-type, but not a set.
- For a function $f : A \to B$, the statement "f is an equivalence" is a mere proposition.

- For a type A and $n \geq -2$, the statement "A is an n-type" is a mere proposition.
- The 2-sphere S^2 is defined to be a higher inductive type with constructors base : S^2 and surf : $\mathrm{refl}_{\mathrm{base}} = \mathrm{refl}_{\mathrm{base}}$. Note that surf is a 2-dimensional path constructor. S^2 is strongly expected not to be n-truncated for any n, but this is not yet proven.

We write $\mathrm{Prop} :\equiv \Sigma(X : \mathcal{U}),\ \text{is-prop}(X)$ for the type of mere propositions. Given $Y : \mathrm{Prop}$ we also write Y the underlying type of Y.

For any type A and $n \geq -1$ we can define the *n-truncation* $\|A\|_n$ of A which is defined to have the same elements as A but all $(n+1)$-dimensional paths equated. We already saw the *propositional truncation*, which is the (-1)-truncation of A, where we identify all elements in A with each other. We write $\|A\|$ for the propositional truncation. The *set-truncation* or 0-truncation identifies all parallel paths in A (two paths are said to be parallel if they have the same type). In general, the n-truncation $\|A\|_n$ can be defined as a HIT [Uni13, Section 7.3]. The type $\|A\|_n$ is an n-type, and it is universal in the sense that functions $A \to B$ where B is an n-type factor through $\|A\|_n$.

A notion related to an n-type is an *n-connected type*, which is a type that has trivial path spaces below dimension n. Formally, A is n-connected if $\|A\|_n$ is contractible. A type which is 0-connected is called *connected* and a type which is 1-connected is called *simply connected*, where the names are taken from the corresponding notions in homotopy theory.

3. Intuition

In this section we will present an intuition about the correctness of the construction. We will also give some properties of one-step truncations.

To give an intuition why this constructions works, consider the type of booleans $\mathbf{2}$ with its two inhabitants $0, 1 : \mathbf{2}$, and take its propositional truncation: $\|\mathbf{2}\|$. Of course, this type is equivalent to the interval (and any other contractible type), but we want to study its structure a little more closely. The path constructor of the propositional truncation ε gives rise to two paths between $|0|$ and $|1|$, namely $\varepsilon(|0|, |1|)$ and $(\varepsilon(|1|, |0|))^{-1}$. Since the resulting type is a mere proposition, these paths must be equal. And indeed, we can explicitly construct a path between them by using the continuity of ε. To do this, we show that for every $x : \|\mathbf{2}\|$ and every $p : |0| = x$ we have $p = \varepsilon(|0|, |0|)^{-1} \cdot \varepsilon(|0|, x)$. We can apply path induction on p, and it is trivial if p is reflexivity. Since the right hand side does not depend on p, this shows that any two elements in $|0| = |1|$ are equal. Note that this is just the proof that any proposition is a set given in [Uni13, Lemma 3.3.4].

Now consider the one-step truncation of the booleans, $\{\mathbf{2}\}$. This is a type with points $a :\equiv f(0)$ and $b :\equiv f(1)$ and four basic paths, as displayed in Figure 1. The path constructor gives two paths from a to b, namely $p :\equiv e(0, 1)$ and $q :\equiv (e(1, 0))^{-1}$. Also, we have two loops $e(0, 0) : a = a$ and $e(1, 1) : b = b$. We can also concatenate these paths to get new paths. Note that the paths p and q are *not* equal. This is because p and q are different path constructors of $\{\mathbf{2}\}$, so we can use the elimination principle of the one-step truncation to send them to any pair of parallel paths. So $p = q$ would imply that all types are sets, contradicting the Univalence Axiom. Applying $\{-\}$ to $\mathbf{2}$ is the first step towards the propositional truncation: all points are equal, but these equalities are not equal themselves.

Now we can take the one-step truncation again to get the type $\{\{\mathbf{2}\}\}$. In this type we have the points $f(a)$ and $f(b)$ and paths $\mathrm{ap}_f(p)$ and $\mathrm{ap}_f(q)$ between them. Now these paths *are* equal, as displayed in Figure 2. If we view the space $\{\{\mathbf{2}\}\}$ as taking the space $\{\mathbf{2}\}$ and adding additional paths between the points, this means we added sufficiently many paths to create a surface between

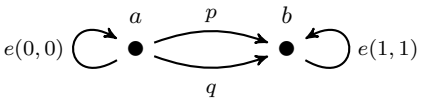

Figure 1. The basic paths in $\{\mathbf{2}\}$

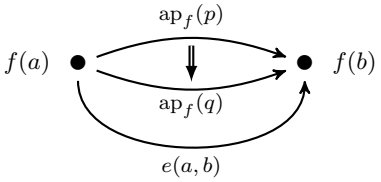

Figure 2. Some paths in $\{\{\mathbf{2}\}\}$.

the paths p and q. The fact that $\mathrm{ap}_f(p)$ and $\mathrm{ap}_f(q)$ are equal is a consequence of the following lemma.

Lemma 3.1. *If $g : X \to Y$ is weakly constant, then for every $x, x' : X$, the function $\mathrm{ap}_g : x = x' \to g(x) = g(x')$ is weakly constant. That is, $\mathrm{ap}_g(p) = \mathrm{ap}_g(q)$ for all $p, q : x = x'$.*

Proof. Let $q : \Pi(x, y : X),\ g(x) = g(y)$ be the proof that g is weakly constant, and fix $x : X$. We first prove that for all $y : X$ and $p : x = y$ we have

$$\mathrm{ap}_g(p) = q(x, x)^{-1} \cdot q(x, y). \tag{3}$$

This follows from path induction, because if p is reflexivity, then $\mathrm{ap}_g(\mathrm{refl}_x) \equiv \mathrm{refl}_{g(x)} = q(x, x)^{-1} \cdot q(x, x)$. The right hand side of (3) does not depend on p, hence ap_g is weakly constant. □

Since f is weakly constant (as the path constructor e shows), we conclude that $\mathrm{ap}_f(p) = \mathrm{ap}_f(q)$. So we see that in $\{\{\mathbf{2}\}\}$ the paths $\mathrm{ap}_f(p)$ and $\mathrm{ap}_f(q)$ are equal. More generally, any pair of parallel paths in A will be equal in $\{A\}$ after applying f. However, in $\{\{\mathbf{2}\}\}$ we also add *new* paths between $f(a)$ and $f(b)$, for example $e(a, b)$ (see Figure 2). This path is not equal to the old paths $\mathrm{ap}_f(p)$ or $\mathrm{ap}_f(q)$. More generally, we have the following proposition. We will not use this proposition for the proof of 4.3.

Proposition 3.2. *For this proposition we assume the Univalence Axiom. Let A be a type.*

1. *For any path $p : a =_A b$ we have $\mathrm{ap}_f(p) \neq e(a, b)$.*
2. *If the type $\|\{A\}\|_1$ is a set, then it is empty. That is,*
$$\text{is-set}(\|\{A\}\|_1) \to \|\{A\}\|_1 \to \mathbf{0}.$$
In particular, $\{A\}$ is not simply connected.
3. *$\|\{A\}\|_0 \simeq \|A\|$, hence $\|\{A\}\|_0$ is a mere proposition. In particular, if A is inhabited, then $\{A\}$ is connected.*

Proof. **Part 1**. Assume that $\mathrm{ap}_f(p) = e(a, b)$. Now define a map $h : \{A\} \to S^1$ by pattern matching as:

- $h(f(a)) :\equiv$ base for all $a : A$;
- $\mathrm{ap}_h(e(a, b)) :=$ loop for all $a, b : A$.

We compute

$$\begin{aligned}\mathrm{refl}_{\mathrm{base}} &= \mathrm{ap}_{\lambda x,\mathrm{base}}(p)\\ &\equiv \mathrm{ap}_{h\circ f}(p)\\ &= \mathrm{ap}_h(\mathrm{ap}_f(p))\\ &= \mathrm{ap}_h(e(a,b))\\ &= \mathrm{loop}.\end{aligned}$$

This is a contradiction. If the universe does not contain the circle, then choose any other type X with a point $x : X$ and a loop $p : x = x$ such that $p \neq \mathrm{refl}_x$. The Univalence Axiom ensures that such a type exists. In this case we can define h similarly, mapping into X.

Part 2. Suppose that $\|\{A\}\|_1$ is an inhabited set. We will construct an element of $\mathbf{0}$. Let $z : \|\{A\}\|_1$ be an inhabitant. We can apply $\|-\|_1$-induction and then $\{-\}$-induction on z. The path constructors are satisfied automatically, since we are proving a mere proposition. This means that we only have to show it for point constructors, hence we may assume that A is inhabited. So assume $x : A$.

We have two inhabitants of $f(x) = f(x)$, namely $e(x, x)$ and $\mathrm{refl}_{f(x)}$. We can use the fact that $\|\{A\}\|_1$ is a set to show that these elements are merely equal. First we use the characterization of path spaces in truncated types [Uni13, Theorem 7.3.12] (using UA) to note:

$$\|f(x) = f(x)\|_0 \simeq \left(|f(a)|_1 =_{\|\{A\}\|_1} |f(a)|_1\right).$$

The right hand side is an equality type in a set, hence it is a mere proposition, and so is the left hand side. Now apply Theorem 7.3.12 again to get:

$$\|e(x,x) = \mathrm{refl}_{f(x)}\| \simeq \left(|e(x,x)|_0 =_{\|f(x)=f(x)\|_0} |\mathrm{refl}_{f(x)}|_0\right).$$

The right hand side is an equality in a mere proposition, hence it is contractible. So the left hand side is also contractible, and in particular inhabited. Since we are proving a mere proposition, we may assume that $e(x, x) = \mathrm{refl}_{f(x)}$. The contradiction follows from part 1.

Part 3. To prove the equivalence, we first prove that $\|\{A\}\|_0$ is a mere proposition. Given $x, y : \|\{A\}\|_0$, we have to show that $x = y$. This is an equality type in a set, hence a mere proposition, so we may apply $\|-\|_0$-induction and then $\{-\}$-induction on both x and y. The remaining goal is to show that for all $a, b : A$ we have $|f(a)| = |f(b)|$. This type is inhabited by $\mathrm{ap}_{|-|}(e(a, b))$.

Now the equivalence $\|\{A\}\|_0 \simeq \|A\|$ is easy, because we know that both types are mere propositions, hence we only need to define maps in both ways. We can define those maps easily by induction. □

We conclude that if A is an inhabited type then $\{A\}$ is connected but not simply connected. So when we apply $\{-\}$ to an inhabited type A we add equalities in such a way that we make every existing two points equal and any two existing parallel paths equal, but we also add new paths which are not equal to any existing paths in A. Hence we have to repeat this ω many times: at every step we kill off the existing higher equality structure, but by doing so we create new higher equality structure. After ω many times we have killed of all the higher structure of A and are left with its propositional truncation.

4. The Main Theorem

Now we will prove that the construction of the propositional truncation works, in the sense that the construction $A \mapsto \{A\}_\infty$ has the same formation, introduction, elimination and computation rules for the propositional truncation.

We recall the definition of $\{A\}_\infty$. Given a type A, we define a sequence $\{A\}_- : \mathbb{N} \to \mathcal{U}$ by

$$\begin{aligned}\{A\}_0 &:\equiv A\\ \{A\}_{n+1} &:\equiv \{\{A\}_n\}\end{aligned} \tag{4}$$

We have map $f_n :\equiv f : \{A\}_n \to \{A\}_{n+1}$ which is the constructor of the one-step truncation. We define $\{A\}_\infty = \mathrm{colim}(\{A\}_-, f_-)$. This is the formation rule of the propositional truncation (note that $\{A\}_\infty$ lives in the same universe as A).

We also easily get the point constructor of the propositional truncation, because that is just the map $i_0 : A \to \{A\}_\infty$. The path constructor $\Pi(x, y : \{A\}_\infty),\ x = y$, i.e. the statement that $\{A\}_\infty$ is a mere proposition, is harder to define. We will postpone this until after we have defined the elimination and computation rules.

The elimination principle — or induction principle — for the propositional truncation is the following statement. Suppose we are given a family of propositions $P : \{A\}_\infty \to \mathrm{Prop}$ with a section $h : \Pi(a : A),\ P(i_0(a))$. We then have to construct a map $k : \Pi(x : \{A\}_\infty),\ P(x)$. To construct k, take an $x : \{A\}_\infty$. Since x is in a colimit, we can apply induction on x. Notice that we construct an element in $P(x)$, which is a mere proposition, so we only have to define k on the point constructors. This means that we can assume that $x \equiv i_n(a)$ for some $n : \mathbb{N}$ and $a : \{A\}_n$. Now we apply induction on n.

If $n \equiv 0$, then we can choose $k(i_0(a)) :\equiv h(a) : P(i_0(a))$.

If $n \equiv \ell + 1$ for some $\ell : \mathbb{N}$, we know that $a : \{\{A\}_\ell\}$, so we can induct on a. The path constructor of this induction is again automatic. For the point constructor, we can assume that $a \equiv f(b)$. In this case we need to define $k(i_{\ell+1}(f(b))) : P(i_{\ell+1}(f(b)))$. By induction hypothesis, we have an element $y : P(i_\ell(b))$. Now we can transport x along the equality $(g_\ell(b))^{-1} : i_\ell(b) = i_{\ell+1}(f(b))$. This gives the desired element in $P(i_{\ell+1}(f(b)))$.

We can write the proof in pattern matching notation:

- $k(i_0(a)) :\equiv h(a)$
- $k(i_{n+1}(f_n(a))) :\equiv (g_n(b))^{-1}_*(k(i_n(b)))$

The definition $k\ (i_0\ a) :\equiv h\ a$ is also the judgmental computation rule for the propositional truncation.

For the remainder of this section we will prove that $\{A\}_\infty$ is a mere proposition. We will need the following lemma.

Lemma 4.1. *Let X be a type with $x : X$. Then the type $\Pi(y : X),\ x = y$ is a mere proposition.*

Proof. To prove that $\Pi(y : X),\ x = y$ is a mere proposition, we assume that it is inhabited and show that it is contractible. Let $f : \Pi(y : X),\ x = y$. From this, we conclude that X is contractible with center x. Now given any $g : \Pi(y : X),\ x = y$. We know that f and g are pointwise equal, because their codomain is contractible. By function extensionality we conclude that $f = g$, finishing the proof. □

To prove that $\{A\}_\infty$ is a mere proposition, we need to show $\Pi(x, y : \{A\}_\infty),\ x = y$. Since $\Pi(y : \{A\}_\infty),\ x = y$ is a mere proposition, we can use the induction principle for the propositional truncation on x, which we have just proven for $\{A\}_\infty$. This means we only have to show that for all $a : A$ we have $\Pi(y : \{A\}_\infty),\ i_0(a) = y$. We do not know that $i_0(a) = y$ is a mere proposition,[1] so we will just use the regular induction principle for colimits on y. We then have to construct two inhabitants of the following two types:

[1] Of course, we do know that it is a mere proposition after we have finished the proof that $\{A\}_\infty$ is a mere proposition.

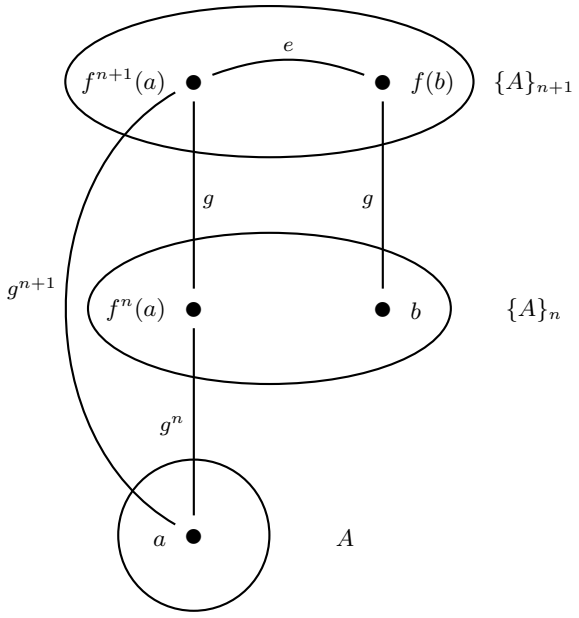

Figure 3. The definition of p. The applications of i and the arguments of the paths are implicit.

1. For the point constructor we need $p(a,b) : i_0(a) = i_n(b)$ for all $a : A$ and $b : \{A\}_n$.
2. We have to show that p respects path constructors:

$$p(a, f(b)) \cdot g(b) = p(a,b). \tag{5}$$

We have a map $f^n : A \to \{A\}_n$ defined by induction on n, which repeatedly applies f. We also have a path $g^n(a) : i_n(f^n(a)) = i_0(a)$ which is a concatenation of instances of g.

We can now define $p(a,b)$ as displayed in Figure 3, which is the concatenation

$$\begin{aligned} i_0(a) &= i_{n+1}(f^{n+1}(a)) && \text{(using } g^{n+1}\text{)} \\ &\equiv i_{n+1}(f(f^n(a))) \\ &= i_{n+1}(f(b)) && \text{(using } e\text{)} \\ &= i_n(b) && \text{(using } g\text{)} \end{aligned}$$

Note that by definition $g^{n+1}(a) \equiv g(f^n(a)) \cdot g^n(a)$, so the triangle on the left of Figure 3 is a definitional equality.

Now we have to show that this definition of p respects the path constructor of the colimit, which means that we need to show (5). This is displayed in Figure 4. We only need to fill the square in Figure 4. To do this, we first need to generalize the statement, because we want to apply path induction. Note that if we give the applications of i explicitly, the bottom and the top of this square are

$$\mathrm{ap}_i(e(f^{n+1}(a), f(b)))$$

and

$$\mathrm{ap}_i(e(f^{n+2}(a), f(f(b)))),$$

respectively. This means we can apply the following lemma to prove this equality.

Lemma 4.2. *Suppose we are given $x, y : \{A\}_n$, $p : x = y$ and $p' : f(x) = f(y)$. Then we can fill the outer square in Figure 5, i.e.*

$$g(x) \cdot \mathrm{ap}_i(p) = \mathrm{ap}_i(p') \cdot g(y).$$

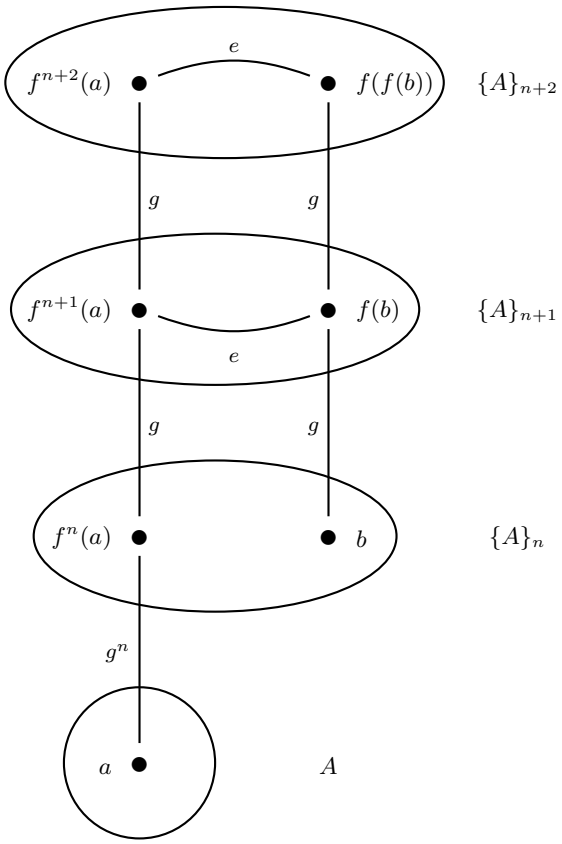

Figure 4. The coherence condition for p. The applications of i and the arguments of the paths are implicit.

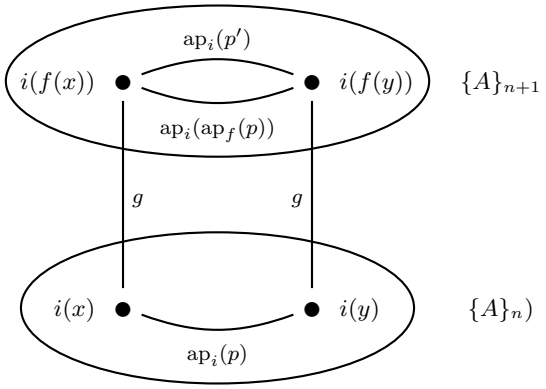

Figure 5. The situation in Lemma 4.2.

Proof. We can fill the inner square of the diagram by induction on p, because if p is reflexivity then the inner square reduces to

$$g(x) \cdot \mathrm{refl}_{i(x)} = \mathrm{refl}_{i(f(x))} \cdot g(x).$$

To show that the two paths in the top are equal, first note that $i_k : \{A\}_k \to \{A\}_\infty$ is weakly constant. To see this, look at Figure 3. The path from $f^n(a)$ to b in that figure gives a proof of $i_n(f^n(a)) = i_n(b)$ which does not use the form of $f^n(a)$, so we also have $i_k(u) = i_k(v)$ for $u, v : \{A\}_k$. Since i_{n+1} is weakly constant, by Lemma 3.1 the function

$$\mathrm{ap}_{i_{n+1}} : f(x) = f(y) \to i_{n+1}(f(x)) = i_{n+1}(f(y))$$

is also weakly constant. This means that the two paths in the top are equal, proving the Lemma. □

We have now given the proof of the following theorem:

Theorem 4.3. *The map $A \mapsto \{A\}_\infty$ satisfies all the properties of the propositional truncation $\|-\|$, including the universe level and judgmental computation rule.*

5. Consequences

As discussed in the introduction, we have the following immediate corollary of Theorem 4.3.

Corollary 5.1.

- *In a type theory with a propositional truncation operation, $\{A\}_\infty \simeq \|A\|$.*
- *In a type theory without propositional truncation operation, we can define it as $\|A\| :\equiv \{A\}_\infty$.*

In HoTT, there are multiple gradations for functions to be constant. We use the terminology from [Shu15].

Definition 5.2. Suppose we are given a function $f : A \to B$.

- f is *weakly constant* if $\Pi(x, y : A),\ f(x) = f(y)$;
- f is *conditionally constant* if it factors through $\|A\|$, i.e. if
$$\Sigma(g : \|A\| \to B),\ \Pi(a : A),\ f(a) = g(|a|);$$
- f is *constant* if $\Sigma(b : B),\ \Pi(a : A),\ f(a) = b$.

It is not hard to see that a constant function is conditionally constant, and that a conditionally constant function is weakly constant. Note that these definitions are not generally mere propositions.

We can also use the universal property of the colimit to get a new universal property for the propositional truncation, which specifies the function space $\|A\| \to B$ not only if B is a mere proposition, but for an arbitrary type B. Recall the definition of $\{A\}_n$ (defined in (4)).

Corollary 5.3. *Let A and B be types.*

1. *We have the following universal property for the propositional truncation:*
$$(\|A\| \to B) \simeq$$
$$(\Sigma(h : \Pi n,\ \{A\}_n \to B),\ \Pi n,\ h_{n+1} \circ f \sim h_n).$$
2. *For a function $k : A \to B$ we have:*
$$(k \text{ is conditionally constant}) \simeq$$
$$(\Sigma(h : \Pi n,\ \{A\}_n \to B),\ (\Pi n,\ h_{n+1} \circ f \sim h_n) \times h_0 \sim k).$$

Proof. The first part is just the universal property of the colimit. The second part follows from the first part, using some basic equivalences about Σ-types. □

We can use our theorem to give a proof of the following.

Lem 3.1	`weakly_constant_ap`
Prop 3.2	`tr_eq_ne_idp`
	`not_inhabited_hset_trunc_one_step_tr`
	`trunc_0_one_step_tr_equiv`
Lem 4.1	in standard HoTT library
Lem 4.2	`ap_f_eq_f`
Th 4.3	`is_hprop_truncX`
Cor 5.1	the definitions below `is_hprop_truncX` define propositional truncation,
	`trunc_equiv`
Cor 5.3	`elim2_equiv`
	`conditionally_constant_equiv`
Cor 5.4	`has_split_support_of_is_collapsible`

Table 1. Names of Theorems in the formalization

Corollary 5.4. *Every* collapsible *type* has split support. *That is, given a weakly constant function $h : A \to A$, then there is a function $\|A\| \to A$.*

Proof. The weakly constant function h gives a function $\tilde{h} : \{A\} \to A$. The HIT $\{-\}$ is functorial (just like all other HITs), so by its functorial action we get a map $\{\tilde{h}\} : \{\{A\}\} \to \{A\}$, which we can compose with $\tilde{h}$ to get a map $\{\{A\}\} \to A$. By induction on n we get a map $k_n : \{A\}_n \to A$. Formally, we define

$$k_0(a) :\equiv a$$
$$k_{n+1}(x) :\equiv \tilde{h}(\{k_n\}(x))$$

However, this sequence of maps does not form a cocone, because the triangles do not commute. (For example for the first triangle we have to show $h(a) = a$ for all a.) But we can easily modify the definition by postcomposing with h. Define $h_n :\equiv h \circ k_n : \{A\}_n \to A$. Now we get a cocone; all triangles commute because h is weakly constant. By Corollary 5.3 we get a map $\|A\| \to A$. □

6. Formalization

All the results in this paper have been formally verified in the proof assistant Lean. The formalization is a single file, available at `https://github.com/fpvandoorn/leansnippets/blob/master/cpp.hlean`. To compile the file, follow these steps:

1. Download this file as `cpp.hlean`.
2. Install Lean via the instructions provided at `http://leanprover.github.io/download/`.
3. Either run `lean cpp.hlean` via your terminal, or open the file `cpp.hlean` in Emacs and execute it using `C-c C-x`.

The injection from the theorems in this paper to the theorems in the formalization is given in Table 1. The formalization closely follows the proof presented here. Actually, a large part the proof was first given in Lean, and found by proving successive goals given by Lean. After the proof was given in Lean, I had to "unformalize" it to a paper proof. So the proof assistant actively helped with constructing the proof. However, when unformalizing the proof, I gained a much better insight in broader picture of the proof, the proof assistant doesn't help very much with that goal.

The formalization heavily uses Lean's *tactic proofs*. In Lean you can give a proof of a theorem either by giving an explicit proof term (as in Agda), or by successively applying tactics to the current goal, changing the goal (often using backwards reasoning), as in Coq. We mainly used tactic proofs, because proofs using tactics are often shorter and quicker to write. This comes at the cost of readability. Tactic proofs are often less readable than their

declarative counterparts, since the proof script is only one part of a "dialogue" between the user and the proof assistant. The other part — the goals given by the proof assistant — are not given in the proof script. However, you can request the goal state at a particular point. To do this, open the file in Emacs, and place the point at the desired location. Now press `C-c C-g` to view the goal. Similarly, you can see the type of a definition by moving the point on it and pressing `C-c C-p`. For more information on how to interact with Lean, see the Lean tutorial [AdMK15].

In Lean, we define the one-step truncation using the "quotient," which is a primitive higher inductive type in Lean. The quotient is the following HIT. Given a type A and a type-valued relation $R : A \to A \to \mathcal{U}$. Then the quotient has two constructors:

- $\iota : A \to \text{quotient}_A(R)$
- $\Pi(x\ y : A),\ R(x, y) \to \iota(x) = \iota(y)$.

The quotient allows us to easily define a large class of HITs. We can define all HITs satisfying the following conditions:

- It has only point and 1-path constructors.
- All constructors are nonrecursive.
- The path constructors don't mention the other path constructors.

In this case, we can take A to be the type defined by the point constructors, and the relation R generated by the path constructors. Although this list may seem restrictive, a lot of HITs still fall into this class. The sequential colimit and one-step truncation fall in this class. Other examples include the circle, suspensions, coequalizers and pushouts.

What's more interesting is that the quotient also defines HITs which do *not* fall in this class. In this paper we have demonstrated that the propositional truncation can be defined just using quotients. Another class of HITs which can be defined using quotients is HITs with 2-constructors. These 2-constructors are equalities between concatenations of 1-constructors. The exact HITs we can form is described in [vD15, section HITs]. This construction allows us to construct HITs such as the torus and the groupoid quotient.

In Lean, the quotient is a primitive notion: the type former, constructors and recursor are constants, and the computation rule for the recursor is added as a computation rule. This is also done for one other HIT, namely n-truncations. Using just these two HITs, we can define all commonly used HITs.

In Coq and Agda, HITs are usually defined using Dan Licata's trick [Lic11]. To define a HIT X using this trick, you first define a normal inductive type Y with the point constructors. Then you add the path constructors of X as constants/axioms to Y and define the induction principle of X using the induction principle for Y. Now the path constructors are inconsistent with the induction principle for Y, so you have to make sure to never accidentally use the induction principle for Y anymore. In Coq and Agda this can be done using private inductive types, which means that the induction principle Y is hidden outside the module where Y is defined. Outside the module it's not visible that the HIT X was defined in an inconsistent way. The disadvantage of this way is that the implementation is inconsistent. This is different in Lean, where we only add two HITs to the type theory, which can be consistently added.

7. Conclusion, Related and Future Work

This construction of the propositional truncation as a colimit of types is a promising method to reduce recursive HITs to non-recursive HITs. It might be possible to use a similar method to construct more general HITs, such as the n-truncation, or more generally, localizations [Shu11]. Another HIT which may be reducable to quotients is the W-suspensions, as defined in [Soj15]. For the n-truncation the construction given here can easily be generalized, but the proof that the resulting colimit is n-truncated does not seem to generalize, and seems to require new ideas.

Corollary 5.4 was already known, and appears in [KECA13, Theorem 3] and [KECA14, Theorem 4.5]. These known proofs show that the type of fixed points of a weakly constant endofunction $A \to A$ is a mere proposition, but there are multiple proofs of this fact. Two of them are given in [KECA14, Lemma 4.1]. Also, Lemma 3.1 already occurs in [KECA13, Proposition 3].

The generalized universal property as written in Corollary 5.3 is new. It is a promising theorem, which simplifies the construction of functions from the propositional truncation to a type which is not truncated. One application of this universal property is already given by Corollary 5.4. This proof doesn't require coming up with a suitable proposition which is an intermediate step between $\|A\|$ and A. The other proofs used as intermediate step the type of fixed points of the endofunction.

Corollary 5.3 is closely related to the main result in [Kra15]. Their main result also gives condition which is equivalent to finding a map $\|A\| \to B$ for an arbitrary type. The advantage of our universal property is that it can be formulated internal to a type theory and that it has been formalized in the proof assistant Lean. Kraus' universal property can only be formulated in a type theory which has certain Reedy limits.

On the other hand, the advantage of Kraus' universal property over the one presented here is that it reduces to a simpler condition when defining a map $\|A\| \to B$ if it's known that B is an n-type. In that case, their condition becomes giving only finitely many higher paths, which can be formulated inside type theory without Reedy limits. Our universal property doesn't simplify given that B is an n-type. It may be possible to modify the colimit, so that the n-th term in the sequence is n-connected. In that case, elimination to an n-type requires only finitely much information, because at some point the cocone becomes trivial.

Another question which involves Corollary 5.3 is whether it is possible to formulate a similar universal property without defining the sequence $\{A\}_n$. This might simplify the universal property.

In Section 3 we have given some properties of the one-step truncation. However, we haven't given an exact characterization. It is not hard to see that the one-step truncation of a set with n elements is a wedge of $n^2 - n + 1$ circles.[2] However, it is not clear whether the structure of the one-step truncation of other types (which are not sets) can be described in more simple terms, for example if $\{\{\mathbf{2}\}\}$ can be described in simpler terms.

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[2] The one-step truncation adds n^2 many paths, and we need to collapse $n-1$ of those paths to reduce the n points to a single point. The result is a wedge of $n^2 - (n-1)$ circles.

[3] available at http://homotopytypetheory.org/2015/07/28/constructing-the-propositional-truncation-using-nonrecursive-hits/

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