Blueprint for the collaborative Formalization Seminar

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2.1 Basic Properties of the Fourier Transform

In this section we record, mostly without proofs, basic statements about the Fourier transform on L^1 functions. Most of these are already formalized in mathlib.

Let $(V, \mu), (W, \nu)$ be vector spaces over $\mathbb{R}$ with a σ-finite measure, E, F be normed spaces over $\mathbb{C}$ and let $L : V \times W \to \mathbb{R}$, $M : E \times F \to \mathbb{C}$ be bilinear maps.

Definition 2.1. *Let $f \in L^1(V, E)$. Its* Fourier transform *(w.r.t. L) is the function $\mathcal{F}f = \hat{f} : W \to E$ given by*

$$\mathcal{F}f(w) := \hat{f}(w) := \int_V e^{-2\pi i L(v,w)} f(v)\, d\mu.$$

The inverse Fourier transform *(w.r.t. L) is similarly defined as*

$$\mathcal{F}^{-1}f(w) = \check{f}(w) := \int_V e^{2\pi i L(v,w)} f(v)\, d\mu.$$

Lemma 2.2. *Let $f \in L^1(V, E)$. Then its Fourier transform $\hat{f}$ is well-defined and bounded. In particular, the Fourier transform defines a map $\mathcal{F} : L^1(V, E) \to L^\infty(V, E)$.*

Proof. Omitted. □

From now on assume that V and W are equipped with second-countable topologies such that L is continuous.

Lemma 2.3. *Let $f \in L^1(V, E)$. Then $\hat{f}$ is continuous.*

Proof. Omitted. □

Lemma 2.4 (Multiplication formula)**.** *Let $f, g \in L^1(V, E)$. Then*

$$\int_W M(\hat{f}(w), g(w))\, d\nu(w) = \int_V M(f(v), \hat{g}(v))\, d\mu(v).$$

Proof. Omitted. □

Lemma 2.5. *Lef $f, g \in L^1(V, E)$, $t \in \mathbb{R}$ and $a, b \in \mathbb{C}$. The Fourier transform satisfies the following elementary properties:*

(i) $\mathcal{F}(af + bg) = a\mathcal{F}f + b\mathcal{F}g$ *(Linearity)*

(ii) $\mathcal{F}(f(x-t)) = e^{-2\pi ity}\mathcal{F}f(y)$ *(Shifting)*

(iii) $\mathcal{F}(f(tx)) = \frac{1}{|t|}\mathcal{F}f(\frac{y}{t})$ *(Scaling)*

(iv) If E *admits a conjugation, then* $\mathcal{F}(\overline{f(x)}) = \overline{\mathcal{F}f(-y)}$ *(Conjugation)*

(v) Define the convolution of f *and* g *w.r.t. a bilinear map* $M : E \times E \to F$ *as*

$$(f *_M g)(w) := \int_V M(f(v), g(w-v))\, d\mu(v).$$

Then $\mathcal{F}(f *_M g) = M(\mathcal{F}f, \mathcal{F}g)$ *(Convolution)*

Proof. Omitted. □

From now on, let V be a finite-dimensional inner product space. We denote this product as ordinary multiplication, and the induced norm as $|\cdot|$.

We now study a family of functions which is useful for later proofs.

Lemma 2.6. *Let* $x \in V$ *and* $\delta > 0$. *Define the* modulated Gaussian

$$u_{x,\delta}(y) : V \to \mathbb{C}, \quad y \mapsto e^{-\delta\pi|y|^2}e^{2\pi ixy}.$$

Its Fourier transform (w.r.t. the inner product) is given by

$$\widehat{u_{x,\delta}}(z) = \delta^{-n/2}e^{-\pi|x-y|^2/\delta} =: K_\delta(x-z).$$

Proof. By choosing an orthonormal basis, wlog we may assume $V = \mathbb{R}^n$. First note $\widehat{u_{x,\delta}}(z-x) = \widehat{u_{0,\delta}}(z)$, so it is enough to consider $x = 0$. Next,

$$\widehat{u_{0,\delta}}(z) = \int_{\mathbb{R}^n} e^{-\pi\delta|y|^2-2\pi iyz}\, dy = \prod_{i=1}^n \int_{\mathbb{R}} e^{-\pi\delta y_i^2-2\pi iy_iz_i}\, dy_i \quad \text{and} \quad K_\delta(-z) = \prod_{i=1}^n \delta^{-1/2}e^{-\pi y_i^2/\delta},$$

hence we may assume $n = 1$. The change of variables $w = \delta^{1/2}y + iz/\delta^{1/2}$ results in

$$\widehat{u_{0,\delta}}(z) = \int_{\mathbb{R}} e^{-\pi\delta y^2-2\pi iyz}\, dy = \delta^{-1/2}e^{-\pi z^2/\delta}\int_{Im(w)=z/\delta^{1/2}} e^{-\pi w^2}\, dw.$$

Contour integration along the rectangle with vertices $(\pm R, 0), (\pm R, iz/\delta^{1/2})$, together with the bound

$$\left|\int_{\pm R}^{\pm R+iz/\delta^{1/2}} e^{-\pi w^2}\, dw\right| \le \frac{|z|}{\delta^{1/2}} \sup_{w\in[R,R+iz/\delta^{1/2}]} |e^{-\pi w^2}| = \frac{|z|}{\delta^{1/2}}e^{-\pi R^2} \xrightarrow{R\to\infty} 0$$

yields

$$\int_{Im(w)=-z/\delta^{1/2}} e^{-\pi w^2}\, dw = \int_{\mathbb{R}} e^{-\pi w^2}\, dw = 1,$$

finishing the proof. □

Lemma 2.7. *Let* $K_\delta(v) = \delta^{-n/2} e^{-\pi|v|^2/\delta}$ *as in Lemma 2.6. This is a* good kernel*, called the* Weierstrass kernel*, satisfying*

$$\int_V K_\delta(x)\, dx = 1 \quad \textit{and} \quad \int_{|x|>\eta} K_\delta(x)\, dx \xrightarrow{\delta\to 0} 0 \quad \textit{for all } \eta > 0.$$

Furthermore, it satisfies the stronger bounds

$$K_\delta(x) \le \delta^{-n/2} \quad \textit{and} \quad K_\delta(x) \le B\delta^{1/2}|x|^{-n-1}$$

for some constant B *independent of* δ*.*

Proof. By choosing an orthonormal basis, wlog we may assume $V = \mathbb{R}^n$. Then these are all straight-forward calculations:

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2/\delta}\, dx = \delta^{n/2} \int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = \delta^{n/2}.$$

$$\int_{|x|>\eta} \delta^{-n/2} e^{-\pi|x|^2/\delta}\, dx = \int_{|x|>\eta/\delta^{1/2}} e^{-\pi|x|^2} \xrightarrow{\delta\to 0} 0.$$

The first upper bound is trivial. For the second one, consider for $r, z \ge 0$ the inequality

$$\Gamma(r+1) = \int_0^\infty e^{-y} y^r\, dy \ge \int_z^\infty e^{-y} y^r\, dy \ge z^r \int_z^\infty e^{-y}\, dy = z^r e^{-z}.$$

Applied to $z = \pi|x|^2/\delta$ and $r = (n+1)/2$, this gives

$$|x|^{n+1} = \delta^{(n+1)/2} \frac{|x|^{n+1}}{\delta^{(n+1)/2}} \le \underbrace{\frac{\Gamma((n+3)/2)}{\pi^{(n+1)/2}}}_{=:B} \delta^{(n+1)/2} e^{\pi|x|^2/\delta},$$

which is equivalent to the second upper bound of the lemma. □

The following technical theorem is used in the proofs of both the inversion formula and Plancherel's theorem.

Theorem 2.8. *Let* $f : V \to E$ *be integrable. Let* K_δ *be the Weierstrass kernel from Lemma 2.7, or indeed any family of functions satisfying the conditions of Lemma 2.7. Then*

$$(K_\delta * f)(x) := \int_V K_\delta(y) f(x-y)\, d\mu(y) \xrightarrow{\delta\to 0} f(x)$$

in the L^1*-norm. If* f *is continuous, the convergence also holds pointwise.*

Proof. Again we may assume $V = \mathbb{R}^n$. Consider the difference

$$\Delta_\delta(x) := (K_\delta * f)(x) - f(x) = \int_{\mathbb{R}^n} (f(x-y) - f(x)) K_\delta(y)\, dy.$$

We prove L^1-convergence first: Take L^1-norms and use Fubini's theorem to conclude

$$\|\Delta_\delta\|_1 \le \int_{\mathbb{R}^n} \|f(x-y) - f(x)\|_1 K_\delta(y)\, dy.$$

For $\varepsilon > 0$ find $\eta > 0$ small enough so that $\|f(x-y) - f(x)\|_1 < \varepsilon$ when $|y|\eta$. Thus

$$\|\Delta_\delta\|_1 \leq \varepsilon + \int_{|y|>\eta} \|f(x-y) - f(x)\| K_\delta(y)\, dy \leq \varepsilon + 2\|f\|_1 \int_{\|y\|>\eta} K_\delta(y)\, dy.$$

By one of the properties in Lemma 2.7, we can choose δ small enough so that the second integral is less than ε, which finishes the proof in this case.

Now assume that f is continuous. Let $d = \delta^{1/2}$ and shorten $g_\delta(x,y) = |f(x-y) - f(x)|K_\delta(y)$.Then

$$|\Delta_\delta(x)| \leq \int_{0<|y|<d} g_\delta(x,y)\, dy + \sum_{k\in\mathbb{N}} \int_{2^k d<|y|<2^{k+1}d} g_\delta(x,y)\, dy.$$

To bound these integrals, consider

$$\varphi(r) = \frac{1}{r^n} \int_{|y|<r} |f(x-y) - f(x)|\, dy.$$

It is easy to see that φ is continuous, bounded, and approaches 0 for $r \to 0$, by continuity of f. Now

$$\int_{0<|y|<d} g_\delta(x,y)\, dy \overset{(*)}{\leq} d^{-n} \int_{0<|y|<d} |f(x-y) - f(x)|\, dy = \varphi(d)$$

and

$$\int_{2^k d<|y|<2^{k+1}d} g\, dy \overset{(*)}{\leq} \frac{Bd}{(2^k d)^{n+1}} \int_{2^k d<|y|<2^{k+1}d} |f(x-y) - f(x)|\, dy \leq 2^{n-k} B\varphi(2^{k+1}d),$$

where for the inequalities labeled $(*)$ we used the upper bounds from Lemma 2.7. Together, we find

$$|\Delta_\delta(x)| \leq \varphi(d) + C \sum_{k\in\mathbb{N}} 2^{-k}\varphi(2^{k+1}d)$$

for $C = \frac{2^n\Gamma((n+3)/2)}{\pi^{(n+1)/2}}$. Say φ is bounded by $M \in \mathbb{R}$ and let $\varepsilon > 0$. Take N large enough such that $\sum_{k\geq N} 2^{-k} < \varepsilon$. Choose δ small enough that $A(2^k d) < \varepsilon/N$ for all $k < N$. Then

$$|\Delta_\delta(x)| \leq \varepsilon/N + (N-1)C\varepsilon/N + C\varepsilon M \leq \varepsilon C(M+1).$$

□

Remark 2.9. *One can drop the continuity assumption and still get pointwise convergence almost everywhere. The proof stays the same, but one focuses on Lebesgue points of f. It takes slightly more work to argue that φ behaves nicely, but the rest of the proof stays the same.*

Theorem 2.10 (Inversion formula)**.** *Let $f : V \to E$ be integrable and continuous. Assume $\hat{f}$ is integrable as well. Then*

$$\mathcal{F}^{-1}\mathcal{F}f = f.$$

Proof. Apply the multiplication formula Lemma 2.4 to $u_{x,\delta}$ and f, and conclude with Theorem 2.8. □

Remark 2.11. *Note that both assumptions are necessary, since $\mathcal{F}^{-1}\mathcal{F}f$ is continuous, and only defined if $\mathcal{F}f$ is integrable.*

Theorem 2.12 (Inversion formula, L^1-version)**.** *Let $f \in L^1(V,E)$. If $\hat{f} \in L^1(V,E)$, then $\mathcal{F}^{-1}\mathcal{F}f = f$.*

Proof. Similar to Theorem 2.10. □

2.2 Plancherel's Theorem and the Fourier Transform on L^2

Let $(V, \cdot)$ be a finite-dimensional inner product space over $\mathbb{R}$ and let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{C}$.

Theorem 2.13 (Plancherel's Theorem)**.** *Suppose that $f : V \to E$ is in $L^1(V, E) \cap L^2(V, E)$ and let $\hat{f}$ be the Fourier transform of f. Then $\hat{f}, \check{f} \in L^2(V, E)$ and*

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} = \|\check{f}\|_{L^2}.$$

Suppose that $f : V \to E$ is in $L^1(V, E) \cap L^2(V, E)$ and let $\hat{f}$ be the Fourier transform of f. Then $\hat{f}, \check{f} \in L^2(V, E)$ and

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} = \|\check{f}\|_{L^2}.$$

Proof. Let $g(x) = f(-x)$ and apply the multiplication formula Lemma 2.4 to $f * g$ and $u_{0,\delta}$:

$$\int_V \widehat{f * g} \cdot u_{0,\delta}(x)\, dx = \int_V (f * g)(x) K_\delta(-x)\, dx \overset{\delta \to 0}{\to} (f * g)(0) = \int_V \langle f(x), f(x) \rangle\, dx = \|f\|_2^2$$

by Theorem 2.8. On the other hand, by Lemma 2.5 the left hand side simplifies to

$$\int_V |\hat{f}(x)|^2 e^{-\delta\pi |x|^2}\, dx \xrightarrow{\delta \to 0} \|\hat{f}\|_2^2$$

by dominated convergence.

Since $\check{f}(x) = \hat{f}(-x)$, the corresponding statements for $\check{f}$ follow immediately from the ones for $\hat{f}$. □

We now want to extend the Fourier transform to $L^2(V, E)$. For this, take a sequence of functions $(f_n)_n \subset L^1(V, E) \cap L^2(V, E)$ such that $f_n \underset{L^2}{\longrightarrow} f$. Such sequences exist:

Lemma 2.14. *$L^1(V, E) \cap L^2(V, E)$ is dense in $L^2(V, E)$.*

Proof. It is well-known that the space of compactly supported continuous functions is dense in every $L^p(V, E)$. Since those are contained in $L^1(V, E) \cap L^2(V, E)$, the claim follows. □

Let $f \in L^2(V, E)$. Plancherel's theorem lets us now approximate a potential $\hat{f}$:

Lemma 2.15. *Let $f \in L^2(V, E)$ and $(f_n)_n \subset L^1(V, E) \cap L^2(V, E)$ a sequence with $f_n \underset{L^2}{\longrightarrow} f$. Then $(\hat{f}_n)_n$ is a Cauchy sequence, hence converges in $L^2(V, E)$.*

Proof.

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|\widehat{f_n - f_m}\|_2 \overset{\text{Plancherel}}{=} \|f_n - f_m\|_2$$

goes to 0 for n, m large, as $(f_n)_n$ is convergent, hence Cauchy. Since $L^2(V, E)$ is complete, $(\hat{f}_n)_n$ converges. □

Definition 2.16. *Let $f \in L^2(V, E)$ and take a sequence $(f_n)_n \subset L^1(V, E) \cap L^2(V, E)$ with $f_n \underset{L^2}{\longrightarrow} f$. Set*

$$\mathcal{F} f := \hat{f} := \lim_{n \to \infty} \widehat{f_n},$$

the limit taken in the L^2-sense.

Lemma 2.17. *This is well-defined: By Lemma 2.15, the limit exists. Further it does not depend on the choice of sequence* $(f_n)_n$*. If* $f \in L^1(V,E) \cap L^2(V,E)$*, this definition agrees with the Fourier transform on* $L^1(V,E)$*.*

Proof. Let $(g_n)_n$ be another sequence approximating f. Then

$$\|\hat{f}_n - \hat{g}_n\|_2 = \|f_n - g_n\| \leq \|f_n - f\| + \|g_n - g\| \to 0.$$

If $f \in L^1(V,E) \cap L^2(V,E)$, we can choose the constant sequence $(f_n)_n = (f)_n$. ☐

Definition 2.18. *Define analogously* $\mathcal{F}^{-1} f := \check{f} := \lim_{n\to\infty} \check{f}_n$*, if* $f_n \underset{L^2}{\longrightarrow} f \in L^2(V,E)$ *with* $(f_n)_n \subset L^1(V,E) \cap L^2(V,E)$*. By the same arguments as above, this is well-defined.*

Corollary 2.19. *Plancherel's Theorem, the inversion formula, and the properties of Lemma 2.5 hold for the Fourier transform on* $L^2(V,E)$ *as well.*

Proof. All of these follow immediately from the definition and the observation, that all operations (norms, sums, conjugation, ...) are continuous. For example, let $f \in L^2(V,E)$ and take an approximating sequence $(f_n)_n$ as before. Then

$$\|\hat{f}\|_2 = \|\lim_{n\to\infty} \hat{f}_n\|_2 = \|\lim_{n\to\infty} \|\hat{f}_n\|_2 = \lim_{n\to\infty} \|f_n\|_2 = \|\lim_{n\to\infty} f_n\|_2 = \|f\|_2.$$

☐

Corollary 2.20. *The Fourier transform induces a continuous linear map* $L^2(V,E) \to L^2(V,E)$*.*

Proof. This follows immediately from Corollary 2.19: Linearity from the L^2-version of Lemma 2.5, and continuity and well-definedness from the L^2-version of Plancherel's theorem. ☐

Chapter 3

Interpolation

3.1 Rietz-Thorin's Interpolation Theorem

Rietz-Thorin's interpolation theorem is a powerful tool to study boundedness of linear operators between complex L^p spaces. Informally, it states that if a linear map T is bounded as an operator $L^{p_0} \to L^{q_0}$ and as an operator $L^{p_1} \to L^{q_1}$, then it must also be a bounded operator $L^p \to L^q$ whenever $\left(\frac{1}{p}, \frac{1}{q}\right)$ is a convex combination of $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$ and $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$.
Since simple functions are contained in all the L^p spaces, and bounded linear operators are continuous, an equivalent formulation may be: given a bounded linear operator from simple functions to functions that are integrable on all sets of finite measure, if we know it can be extended to bounded linear operators $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$, then it can also be extended $L^p \to L^q$ with p and q as above.

Before we start, let us recall the maximum modulus principle from complex analysis. There are various statements of this in Lean, see the dedicated Mathlib page.

Theorem 3.1. *Let U be a connected open set in a complex normed space E. Let $f : E \to F$ be a function that complex differentiable on U and continuous on $\bar{U}$.*
If $|f(z)|$ takes its maximum on a point $u \in U$, then it must be constant on $\bar{U}$.

Proof. Already formalized in Mathlib, along with several variants. □

Lemma 3.2. *Let S be the strip $S := \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$. Let $f : \overline{S} \to \mathbb{C}$ be a function that is holomorphic on S and continuous and bounded on $\overline{S}$.*
Assume M_0, M_1 are positive real numbers such that for all values of y in $\mathbb{R}$, we have

$$|\phi(iy)| \leq M_0 \qquad |\phi(1 + iy)| \leq M_1$$

i.e., the absolute values of the function on the lines $\{\operatorname{Re} z = 0\}$ and $\{\operatorname{Re} z = 1\}$ are bounded by M_0 and M_1 respectively.
Then, for all $0 \leq t \leq 1$ and for all real values of y, we have

$$|\phi(t + iy)| \leq M_0^{1-t} M_1^t$$

Proof.
If $|\phi|$ is constant, everything holds trivially by setting M_0 and M_1 to be the value of $|\phi|$ at a point. Assume $|\phi|$ non-constant.

- **Case 1**: assume $M_0 = M_1 = 1$, and $\sup_{0 \le x \le 1} |\phi(x+iy)| \to 0$ when $|y| \to \infty$.
Let M be the supremum of $|\phi(z)|$ on $\bar{S}$. Since the function is non-constant, we have $M > 0$.
Let $\{z_n\}$ be a sequence of points in S such that $|\phi(z_n)|$ converges to M.
Since we assumed the absolute value of ϕ goes to zero as $|y|$ goes to infinity, all points where $|\phi(z)| > M - \epsilon$ must be in some rectangle around zero, i.e. the sequence z_n must be bounded.
Hence, there must be a converging subsequence of z_n to a point $z^* \in \bar{S}$.
By maximum modulus principle, z^* must be on the boundary δS, so it must have real part 0 or 1. Hence, by assumption, $|\phi(z^*)| \le 1$, and by construction $|\phi(z)| \le 1$ for all $z \in \bar{S}$, which is what we wanted to show.

- **Case 2**: only assume $M_0 = M_1 = 1$.
For $\epsilon > 0$, define
$$\phi_\epsilon(z) = \phi(z) e^{\epsilon(z^2-1)}$$
If the real part of z is 0, then $z = iy$ and
$$|\phi_\epsilon(z)| = |\phi(z) e^{\epsilon(-y^2-1)}| \le |\phi(z)| \cdot 1 \le 1$$
If the real part of z is 1, then $z = 1 + iy$ and
$$|\phi_\epsilon(z)| = |\phi(z) e^{\epsilon(1-y^2+2iy-1)}| = |\phi(z) e^{\epsilon(-y^2+2iy)}| = |\phi(z) e^{\epsilon(-y^2)}| \le |\phi(z)| \cdot 1 \le 1$$
Moreover,
$$|\phi_\epsilon(x+iy)| \le |\phi(x+iy)| \cdot |e^{\epsilon(z^2-1)}| = |\phi(x+iy)| \cdot |e^{\epsilon(x^2-1-y^2+2ixy)}| = |\phi(x+iy)| \cdot |e^{\epsilon(x^2-1-y^2)}|$$
Hence, for $0 \le x \le 1$ and $|y| \to \infty$, we have that both factors go to zero.
Thus ϕ_ϵ satisfies the hypotheses of case 1, so $|\phi_\epsilon| \le 1$ on the whole strip.
Now, we have pointwise that
$$\lim_{\epsilon \to 0} \phi_\epsilon(z) = \lim_{\epsilon \to 0} \phi(z) e^{\epsilon(z^2-1)} = \phi(z)$$
Hence, for $\epsilon \to 0$, we have $|\phi_\epsilon(z)| \to |\phi(z)|$. Thus,
$$|\phi(z)| = \lim_{\epsilon \to 0} |\phi_\epsilon(z)| \le 1$$
which is what we wanted to show.

- **General case**
If M_0 and M_1 are any two positive real numbers, define
$$\tilde{\phi}(z) = M_0^{z-1} M_1^{-z} \phi(z)$$
Recall that, for $a \in \mathbb{R} \setminus \{0\}$, we have
$$|a^{b+ic}| = |a^b|$$
Hence, if the real part of z is 0, we have
$$|\tilde{\phi}(z)| \le |M_0^{-1}| \cdot |M_1^0| \cdot |\phi(z)| \le \frac{1}{M_0} \cdot M_0 = 1$$

And if the real part of z is 1, we have

$$|\tilde{\phi}(z)| \le |M_0^0| \cdot |M_1^{-1}| \cdot |\phi(z)| \le \frac{1}{M_1} \cdot M_1 = 1$$

From the previous case, we obtain that for arbitrary z in the strip

$$|\tilde{\phi}(z)| \le 1$$

Now, write $z = t + iy$ and unroll the definition of $\tilde{\phi}$ to obtain

$$|M_0^{t-1+y} M_1^{-t-iy} \phi(z)| \le 1$$

The left-hand side is equal to

$$M_0^{t-1} M_1^{-t} |\phi(z)|$$

So we obtain

$$|\phi(z)| \le M_0^{1-t} M_1^t$$

which is exactly what we wanted.

□

Lemma 3.3. *Let p and q be real conjugate exponents. Let f be measurable. Then*

$$\|f\|_{L^q} = \sup_{\|g\|_{L^p} \le 1,\ g \text{ simple}} \|fg\|_{L^1}.$$

In particular, if the right hand side formula is finite, $f \in L^q$.

Proof. That

$$\sup_{\|g\|_{L^p} \le 1,\ g \text{ simple}} \|fg\|_{L^1} \le \|f\|_{L^q}$$

follows from Hölder's inequality.

The other direction is trivial when $\|f\|_{L^q} = 0$. Suppose $\|f\|_{L^q} \ne 0$. Set

$$g = \frac{|f|^{p-1}}{\|f\|_{L^p}^{p-1}}.$$

Then $\|g\|_{L^p} = 1$.

$$\begin{aligned} \|f\|_{L^q} &= \|fg\|_{L^1} \\ &= \sup_{n \in \mathbb{N}} \|fg_n\|_{L^1} && (1) \\ &\le \sup_{\|g\|_{L^p} \le 1,\ g \text{ simple}} \|fg\|_{L^1} && (2) \end{aligned}$$

In (1) and (2), g_n is a monotone sequence of simple function approximating g from below, whose existence is basic real analysis. □

Lemma 3.4. *Let f be measurable and the measure μ be σ-finite. Then*

$$\|f\|_{L^\infty} = \sup_{\|g\|_{L^1} \le 1,\ g \text{ simple}} \|fg\|_{L^1}.$$

Proof. That

$$\sup_{\|g\|_{L^1}\le 1,\ g \text{ simple}} \|fg\|_{L^1} \le \|f\|_{L^\infty}$$

follows from Hölder's inequality.

Suppose $M := \sup_{\|g\|_{L^1}\le 1,\ g \text{ simple}} \|fg\|_{L^1} < \|f\|_{L^\infty}$. Then $\{x||f(x)| \ge M\}$ has positive measure. Since μ is σ-finite, we have a subset $B \subset \{x||f(x)| \ge M\}$ with finite positive measure by a classical lemma. Now set

$$h := \mu(B)^{-1}\chi_B.$$

Then we have

$$\begin{aligned} M &= \int_B \mu(B)^{-1} M d\mu \\ &< \int_B \mu(B)|f| d\mu \\ &= \|fh\|_{L^1}. \end{aligned}$$

But $\|h\| = 1$ and h is simple, so $\|fh\|_{L^1} \le M$, contradiction.

□

As a last step towards proving the theorem, let us recall a consequence of Hölder's inequality, which will only really be substantial in a corner case of our proof.

Lemma 3.5. *Let (X, μ) be a measure space and $0 < p_0 < p_1 \le \infty$, $0 \le p \le \infty$. Assume we have $t \ge 0$ such that*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$$

Let $f : X \to \mathbb{C}$ be a measurable function. Then

$$\|f\|_{L^p} \le \|f\|_{L^{p_0}}^{1-t} \|f\|_{L^{p_1}}^{t}$$

In particular, the f in $L^{p_0}(X) \cap L^{p_1}(X)$, then f is in $L^p(X)$.

Proof. This is just a version of Hölder's inequality, but in order to apply it, we should rule out some trivial cases.

First note that the assumption guarantees that $0 < p_0 \le p \le p_1 \le \infty$ and $0 \le t \le 1$.

When $t = 0$ or $t = 1$, the inequality holds trivially. Hence we may assume $t \neq 0$ and $t \neq 1$. In this case $p < p_1 \le \infty$.

Now we can rearrange the equality into

$$\frac{1}{\frac{p_0}{(1-t)p}} + \frac{1}{\frac{p_1}{tp}} = 1.$$

Using Hölder's inequality, we have

$$
\begin{aligned}
\|f\|_{L^p} &= (\int |f|^p d\mu)^{1/p} \\
&= (\int |f|^{(1-t)p} |f|^{tp} d\mu)^{1/p} \\
&\leq ((\int |f|^{(1-t)p \frac{p_0}{(1-t)p}} d\mu)^{\frac{(1-t)p}{p_0}} (\int |f|^{tp \frac{p_1}{tp}} d\mu)^{\frac{tp}{p_1}})^{1/p} \\
&= (\int |f|^{p_0} d\mu)^{\frac{(1-t)}{p_0}} (\int |f|^{p_1} d\mu)^{\frac{tp}{p_1}} \\
&= \|f\|_{L^{p_0}}^{1-t} \|f\|_{L^{p_1}}^{t}
\end{aligned}
$$

□

Theorem 3.6. *Let (X, μ) and (Y, ν) be measure spaces and consider all L^p spaces to be complex valued.*
Suppose T is a linear map $L^{p_0}(X) + L^{p_1}(X) \to L^{q_0}(Y) + L^{q_1}(Y)$ that restricts to bounded operators $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$. Let M_0, M_1 be the respective bounds, i.e.,

$$
\begin{cases} ||Tf||_{L^{q_0}} \leq M_0 ||f||_{L^{p_0}} \\ ||Tf||_{L^{q_1}} \leq M_1 ||f||_{L^{p_1}} \end{cases}
$$

Then, for any pair (p, q) such that there is a t in $[0, 1]$ for which

$$
\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}
$$

we have that the operator is bounded $L^p \to L^q$, and in particular

$$
||Tf||_{L^q} \leq M_0^{1-t} M_1^t ||f||_{L^p}
$$

Proof. For a valid choice of p, q, note that we both need to show Tf is in L^q and a bound on the L^q norm of Tf. Let q' be the conjugate exponent of q. By Lemma **??** for Tf, we need a bound of the form

$$
\sup_{||g||_{L^{q'}} \leq 1, \quad g \text{ simple}} \left| \int T(f) g \right| \leq M ||f||_{L^p}
$$

- **Case 1**: assume $p < \infty$ and $q > 1$, and let q', q_0^1, q_1' be the conjugate exponents of q, q_0, q_1 respectively.
  - **Subcase a:** assume $f = \sum_i a_i \chi_{E_i}$ is a simple function (finite sum with E_i disjoint of finite measure).
    Let $g = \sum_j b_j \chi_{F_j}$ be a simple function. By writing f as $||f||_{L^p} \cdot \frac{f}{||f||_{L^p}}$ and using linearity of T and integrals, it suffices to prove the above inequality when $||f||_{L^p} = 1$.
    We want to apply the three lines lemma to an appropriate function. Define

$$
\gamma(z) = p \left(\frac{1-z}{p_0} + \frac{z}{p_1} \right) \qquad f_z = |f|^{\gamma(z)} \cdot \frac{f}{|f|}
$$

$$\delta(z) = q'\left(\frac{1-z}{q_0'} + \frac{z}{q_1'}\right) \qquad g_z = |g|^{\delta(z)} \cdot \frac{g}{|g|}$$

Observe that for t as in the statement of the theorem, we have by definition that $\gamma(t) = 1$, hence $f_t = f$.
Moreover, if $\mathrm{Re}(z) = 0$, we have that $\mathrm{Re}(\gamma(z)) = \frac{p}{p_0}$, and hence

$$||f_z||_{L^{p_0}} = \left(\int |f_z|^{p_0}\right)^{\frac{1}{p_0}} = \left(\int ||f|^{\gamma(z)}|^{p_0}\right)^{\frac{1}{p_0}} = \left(\int |f|^{\frac{p}{p_0}\cdot p_0}\right)^{\frac{1}{p_0}} = (||f||_{L^p}^p)^{\frac{1}{p_0}} = 1^{\frac{1}{p_0}} = 1$$

If $\mathrm{Re}(z) = 1$, we have $\mathrm{Re}(\gamma(z)) = \frac{p}{p_1}$, and the exact same computation replacing p_0 with p_1 now shows that $||f_z||_{L^{p_1}} = 1$.
Similarly, one shows that

$$g_t = g \qquad ||g_z||_{L^{q_0'}} = 1 \text{ if } \mathrm{Re}(z) = 0 \qquad ||g_z||_{L^{q_1'}} = 1 \text{ if } \mathrm{Re}(z) = 1$$

Now, we want to apply the three lines lemma to the function

$$\phi(z) := \int (Tf_z)g_z$$

Since f and g are simple and given by the expressions above, we can explicitly write f_z and g_z as

$$f_z = \sum_i |a_i|^{\gamma(z)} \frac{a_i}{|a_i|} \chi_{E_i} \qquad g_z = \sum_j |b_j|^{\delta(z)} \frac{b_j}{|b_j|} \chi_{F_j}$$

(here, we use that the E_i (respectively F_j) are disjoint, so for every point in the domain there is at most one of the E_i covering it).
So, expanding everything by linearity of T and integrals, we obtain

$$\phi(z) = \sum_{i,j} |a_i|^{\gamma(z)} \frac{a_i}{|a_i|} |b_j|^{\delta(z)} \frac{b_j}{|b_j|} \int T(\chi_{E_i})\chi_{F_j}$$

This only depends on z holomorphically in terms of the exponents of the $|a_i|$ and $|b_j|$, so it is a holomorphic function on the strip S in the three lines lemma and it is continuous on S.
It it also bounded on $\bar{S}$. In fact, we wrote ϕ as a finite sum, so we only need to show each summand is bounded. Since the real part of z is between 0 and 1, the terms $|a_i|^{\gamma(z)}$ and $|b_j|^{\delta(z)}$ have bounded norms. Finally, recall that Hölder's inequality states that $||fg||_1 \leq ||f||_p ||g||_1$ for conjugate exponents p, q. Hence,

$$\left|\int T(\chi_{E_i})\chi_{F_j}\right| \leq ||T(\chi_{E_i})\chi_{F_j}||_{L^1} \leq ||T(\chi_{E_i})||_{L^{p_0}} \cdot ||\chi_{F_j}||_{p_0'} \leq M_0 \mu(E_i)\mu(F_j)$$

which is bounded. Moreover, if $\mathrm{Re}(z) = 0$, since $||f_z||_{L^{p_0}} = ||g_z||_{L^{q_0'}} = 1$, we have

$$|\phi(z)| \leq \int |(Tf_z)g_z| \leq ||Tf_z||_{L^{q_0}} \cdot ||g_z||_{L^{q_0'}} \leq M_0 \cdot ||g_z||_{L^{q_0'}} \leq M_0$$

Similarly, if $\mathrm{Re}(z) = 1$, we obtain

$$|\phi(z)| \leq \int |(Tf_z)g_z| \leq ||Tf_z||_{L^{q_1}} \cdot ||g_z||_{L^{q_1'}} \leq M_1 \cdot ||g_z||_{L^{q_1'}} \leq M_1$$

Thus, applying the three lines lemma to $\phi(z)$ yields that

$$|\phi(t+yi)| \leq M_0^{1-t} M_1^t$$

In particular, this holds for $y = 0$, but now

$$\phi(t) = \int (Tf_t) g_t = \int (Tf) g$$

So we have

$$\left| \int (Tf) g \right| \leq M_0^{1-t} M_1^t$$

which is exactly what we wanted to show.

- **Subcase b:** Now, let f be any function in L^p. By density of simple functions, approximate f by a sequence f_n of simple functions with $||f_n - f||_{L^p} \to 0$.
  By the previous case, we have $||Tf_n||_{L^q} \leq M||f_n||_{L^p}$. In particular, the sequence $\{Tf_n\}$ is Cauchy in L^q, since

$$||Tf_m - Tf_n||_{L^q} = ||T(f_m - f_n)||_{L^q} \leq M||f_m - f_n||_{L^p}$$

  and the original sequence is Cauchy. By completeness, the $\{Tf_n\}$ converge in L^q, in particular the L^q norm of the limit is the limit of the L^q norms, which is less than $M||f||_{L^p}$. Hence, it suffices to show that the sequence $\{Tf_n\}$ converges almost everywhere to Tf.
  Write $f = f^U + f^L$ with

$$f^U := \begin{cases} f(x) & \text{if } |f(x)| \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad f^L := \begin{cases} f(x) & \text{if } |f(x)| < 1 \\ 0 & \text{otherwise} \end{cases}$$

  and similarly $f_n = f_n^U + f_n^L$.
  Modulo reordering them, assume $p_0 \leq p_1$, so we have $p_0 \leq p \leq p_1$. Since $f \in L^p$, f^U must be in L^{p_0} and f^L in L^{p_1}. Similarly, since $f_n \to f$ in L^p, we have $f_n^U \to f^U$ in L^{p_0} and $f_n^L \to f^L$ in L^{p_1}.
  By the assumptions of boundedness of L

$$Tf_n^U \to Tf^U \text{ in } L^{q_0} \qquad Tf_n^L \to Tf^L \text{ in } L^{q_1}$$

  Modulo extracting subsequences, we can assume that the convergence is almost everywhere, so that almost everywhere

$$Tf_n(x) = Tf_n^U(x) + Tf_n^L(x) \to Tf^U(x) + Tf(x) = Tf(x)$$

  which is what we wanted to show.

- **Case 2:** $p = \infty$ or $q = 1$.
  If $p = \infty$, we must also have $p_0 = p_1 = \infty$, thus we have

$$||Tf||_{L^{q_0}} \leq M_0||f||_{L^\infty} \qquad ||Tf||_{L^{q_1}} \leq M_1||f||_{L^\infty}$$

  Applying Lemma 3.5 with Tf, q, q_0, q_1, we obtain

$$||Tf||_{L^q} \leq ||Tf||_{L^{q_0}}^{1-t} ||Tf||_{L^{q_1}}^t \leq M_0^{1-t} M_1^t ||f||_{L^\infty}$$

which is what we wanted.

If $p < \infty$ and $q = 1$, then (since they must be at least 1 by definition of L^p spaces) we have that q_0 and q_1 must also both be 1 (for example, since $\frac{1}{q}$ is a convex combination of the other two reciprocals, the largest one must be 1, and from that rearranging terms shows the other one is 1). In this case, take $g_z = g$ for all z and repeat the proof above (note:isn't this what already happens if we do not consider this case separately?).

□

3.2 Applications of Rietz-Thorin's Interpolation Theorem

3.2.1 Hausdorff-Young inequalities

Lemma 3.7. *Let $X = [0, 2\pi]$ with normalized Lebesgue measure $\frac{d\theta}{2\pi}$ and let $Y = \mathbb{Z}$ with counting measure.*
Consider the operator $T : f \mapsto \{a_n\}_{n\in\mathbb{Z}}$ where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$$

For $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$||Tf||_{L^q} \le ||f||_{L^p}$$

Proof. Observe that we may simply regard T as an operator $L^1([0, 2\pi]) \to L^\infty(\mathbb{Z})$ since $L^2([0, 2\pi]) \subseteq L^1([0, 2\pi])$ (compact domain, bound with maximum), and $L^2(\mathbb{Z}) \subseteq L^\infty(\mathbb{Z})$. Note that the claim corresponds (unless q is infinity) to the inequality

$$\left(\sum_{n\in\mathbb{Z}} |a_n|^q\right)^{1/q} \le \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^p d\theta\right)^{1/p}$$

For $p_0 = 2$ (thus $q_0 = 2$), this is Parseval's identity (see tsum_sq_fourierCoeff).
For $p_1 = 1$ (thus $q_1 = \infty$), we can check it directly. Since

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$$

we have

$$|a_n| \le \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)e^{-in\theta}|d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|d\theta = ||f||_{L^1}$$

So

$$||Tf||_\infty = \sup_n |a_n| \le ||f||_{L^1}$$

Applying Rietz-Thorin's theorem, we obtain that the claim holds whenever we can find a $t \in [0, 1]$ such that

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

Substituting $p_0 = 2$, $p_1 = 1$, $q_0 = 2$, $q_1 = \infty$,

$$\frac{1}{p} = \frac{1-t}{2} + t \qquad \frac{1}{q} = \frac{1-t}{2}$$

Now

$$\frac{1}{p} = \frac{1+t}{2} \ \Rightarrow \ p = \frac{2}{1+t}$$

which for $t \in [0, 1]$ ranges from 1 to 2.
Moreover, we have

$$\frac{1}{p} + \frac{1}{q} = \frac{1+t}{2} + \frac{1-t}{2} = 1$$

i.e., p and q are conjugate exponents.
This completes the proof. □

Now, we want to obtain a "dual" inequality to the previous one. For this, we consider an operator $T' : L^2(\mathbb{Z}) \to L^2([0, 2\pi])$ in the opposite direction compared to the previous lemma

$$T'(\{a_n\}_{n\in\mathbb{Z}}) := \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

The operator is defined on any $L^p(\mathbb{Z})$ for $p \leq 2$, since $L^p(\mathbb{Z}) \subseteq L^2(\mathbb{Z})$. Note that the target expression is indeed in $L^2([0, 2\pi])$ again.

Lemma 3.8. *For $1 \leq p \leq 2$ and q conjugate exponent to p, we have*

$$||T'\{a_n\}||_{L^q} \leq ||\{a_n\}||_{L^p}$$

Proof. This is similar to the previous corollary. Parseval's identity gives the case $p_0 = q_0 = 2$. For the case $p_1 = 1$, $q_1 = \infty$, again

$$\left|\sum_{n\in\mathbb{Z}} a_n e^{in\theta}\right| \leq \sum_{n\in\mathbb{Z}} |a_n e^{in\theta}| = \sum_{n\in\mathbb{Z}} |a_n| = ||\{a_n\}||_{L^1}$$

i.e.

$$||T'\{a_n\}||_\infty = \sup_{\theta\in[0,2\pi]} \left|\sum_{n\in\mathbb{Z}} a_n e^{in\theta}\right| \leq ||\{a_n\}||_{L^1}$$

As before, applying Rietz-Thorin's interpolation theorem concludes the proof. □

As a remark, if $f = T'\{a_n\}$, then the $\{a_n\}$ are the Fourier coefficients of f, yielding (when $p \neq 1$) the inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^q d\theta\right)^{1/q} \leq \left(\sum_{n\in\mathbb{Z}} |a_n|^p\right)^{1/p}$$

3.2.2 Extending the Fourier transform

The Rietz-Thorin interpolation theorem also allows us to extend the Fourier transform defined in the previous chapter to a bigger domain.
Let V be a finite dimensional real inner product space and E be a normed complex space. As in Definition 2.1 define the Fourier transform on simple functions f via the expression

$$\mathcal{F}(f)(w) = \int e^{-2\pi i \langle v,w \rangle} f(v)$$

We have shown $\mathcal{F}$ extends to a bounded linear operator $L^1 \to L^\infty$, and to a bounded linear operator $L^2 \to L^2$ (see Theorem 2.20).
By Rietz-Thorin interpolation theorem, it can be uniquely extended to bounded linear operators $L^p \to L^q$ whenever $1 \leq p \leq 2$ and q is conjugate to p.

Chapter 4

Distributions

Laurent Schwartz introduced the notion of distributions two talk about generalized solutions to differential equations. For his work he got the fields medal.

4.1 Space of Distributions

Definition 4.1. *$\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ is the set of test functions together with a topology determined by its converging sequences : $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ if*

- *there exists a compact subset $K \subset \Omega$ such that* $\mathrm{Supp}(\phi_n) \subset K$.
- *For all multiindices α we have $\partial_\alpha \phi_n \to \partial_\alpha \phi$ in uniformly.*

Convention: if $x \in \mathbb{R}^d \setminus \Omega$, then $\phi(x) := 0$.
$\mathcal{D}'(\Omega)$ is the topological dual space, i.e. the space of continuous linear functionals $D(\Omega) \to \mathbb{C}$ with the weak--convergence, i.e. pointwise convergence.*

Remark 4.2. *A notion of converging sequence on a set X is*

- *The constant function on x converges to x*
- *A sequence converges to x iff any subsequence has a subsequence converging to x.*

Note, that any subsequence of a converging sequence converges to the same limit. It induces a closure operator on X (for $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ you use, that if a sequence in $A \cup B$ converges to x, then it has a subsequence (converging to x) lying in A or in B).

Example 4.3. *Every locally integrable function $f \in L^1_{loc}(\Omega)$ gives us a distribution $\Lambda f \in D'(\Omega)$ defined by*

$$\Lambda f(\phi) = \int_\Omega f(x)\phi(x)\ \mathrm{d}x$$

which is well-defined because ϕ has compact support.

Example 4.4. *Let μ be a Radon measure on Ω (or more generally a signed Borel measure which is finite on compact subsets of Ω). Then it defines a distribution:*

$$T_\mu(\phi) = \int_\Omega \phi\ \mathrm{d}\mu$$

Proof. As Borel sets are μ-measurable, every continuous function is μ-measurable. Let $K := \operatorname{Supp} \phi \subset \Omega$. Because $\mu(K) < \infty$ and $\max \phi(K) < \infty$ it follows that ϕ is μ-integrable. Linearity follows from linearity of the integral. If $\phi_j \to \phi_*$ unifomrly, then $\operatorname{Supp} \phi_j \subset K$ for all j and $T_\mu \phi_j \to \Lambda_\mu \phi_*$. □

We have the following important special case:

Example 4.5 (Dirac-δ)**.** *We have $\delta \in D'(\Omega)$ given by*

$$\delta(\phi) := \phi(0)$$

4.1.1 Convolution

Notation 1. *For $\phi \in D$ define $\phi^R \in D$ as $\phi^R(x) = \phi(-x)$ and for $x \in \mathbb{R}^d$ we have the shift $\tau_x(\phi) \in D$ given by $\tau_x(\phi)(y) = \phi(y-x)$.*

Example 4.6. *For $f \in L^1_{loc}(\Omega), g \in D(\Omega)$ we have*

$$(f * g)(x) = \Lambda f(\tau_x(\psi^R))$$

Proposition 4.7. *Let $F \in \mathcal{D}'(\Omega), \psi \in D$. The following two distributions coincide:*

1. *The distribution determined by the smooth function $x \mapsto F(\tau_x(\psi^R))$.*
2. *The distribution $\phi \mapsto F(\psi^R * \phi)$.*

We write this distribution as $F * \psi$.

Proof. $\zeta := \psi^R$ The function $x \mapsto F(\tau_x(\psi^R))$ is smooth :

- It is continuous: If $x_n \to x$, then $\tau_x(\zeta) - \tau_{x_n}(\zeta) \to 0$ uniformly and the same holds for partial derivatives. Now F is continuous hence $F(\tau_\bullet(\psi^R))$ is continuous and F preserves the difference quotients, hence it will be also smooth.
- The two distributions coincide:

$$\begin{aligned} F(\psi^R * \phi) &= F\left(\int (\tau_x \psi^R(\bullet)\phi(x) \; \mathrm{d}x)\right) \\ &\overset{!}{=} \int F(\tau_x \psi^R(\bullet)\phi(x)) \; \mathrm{d}x \\ &= \int F(\tau_x \psi^R)\phi(x) \; \mathrm{d}x \\ &= \Lambda(F(\tau_\bullet \psi^R))(\phi) \end{aligned}$$

  Where we are allowed to pull out the integral by 4.8

□

Lemma 4.8. *Let $\phi \in C_c^\infty(\Omega \times \Omega)$. Then for any $F \in D'(\Omega)$ we have*

$$F\left(\int_\Omega \phi(x, \_) \; \mathrm{d}x\right) = \int_\Omega F(\phi(x, \_)) \; \mathrm{d}x$$

Proof. Consider $S_\varepsilon \in D$ defined by

$$S_\varepsilon^\phi(y) = \varepsilon^d \sum_{n\in\mathbb{Z}^d} \phi(n\varepsilon, y)$$

which is a finite sum as ϕ has compact support. Then for all $\phi \in C_c^\infty(\Omega \times \Omega)$ one has a limit in $\mathcal{D}$

$$\int_\Omega \phi(x, \_) \,\mathrm{d}x = \lim_{\varepsilon\to 0} S_\varepsilon^\phi$$

Hence by continuity of F

$$\begin{aligned} F(\int_\Omega \phi(x, \_) \,\mathrm{d}x) &= F(\lim_{\varepsilon\to 0} S_\varepsilon^\phi) \\ &= \lim_{\varepsilon\to 0} F(S_\varepsilon^\phi) \\ &= \lim_{\varepsilon\to 0} \varepsilon^d \sum_{n\in\mathbb{Z}^d} F(\phi(n\varepsilon, \_)) \\ &= \lim_{\varepsilon\to 0} S_\varepsilon^{F\circ\phi} \\ &= \int_\Omega F(\phi(x, \_)) \,\mathrm{d}x \end{aligned}$$

□

Using the first definition we learn the following two things

Example 4.9. *From the description 4.6: Writing $\Lambda f * g$ is unambiguous.*

Example 4.10. *We have $\delta_0 * f = \Lambda f$ for all $f \in D$.*

Lemma 4.11. *Convolution $F * \psi$ is continuous in both variables.*

Proof. Continuity in the distribution variable is clear by pointwise convergence.
For the continuity in the test function variable one uses that convolution with a fixed test function is a continuous function $\mathcal{D} \to \mathcal{D}$ and distributions are continuous. □

Proposition 4.12. *There exists a sequence $\psi_n \in C_c^\infty(\Omega)$ such that $\Lambda\psi_n \to \delta_0$ in $D'(\Omega)$.*

Proof. Fix some $\psi \in D$ with $\int \psi(x) \,\mathrm{d}x = 1$. Define $\psi_n(x) := n^d\psi(nx)$. Then

$$(\Lambda\psi_n - \delta_0)(\phi) = \int n^d\psi(nx)\phi(x) \,\mathrm{d}x - \phi(0) = \int \psi(x) \cdot (\phi(x/n) - \phi(0)) \,\mathrm{d}x \to 0$$

where we used that $\phi(x/n) - \phi(0) \to 0$ uniformly.

□

The following is how we view $L^1_{loc}(\Omega) \subset D'(\Omega)$.

Corollary 4.13. *Let $f, g \in L^1_{loc}(\Omega)$ such that $\Lambda_f = \Lambda_g$. Then $f = g$ almost everywhere.*

Proof. We have $0 = \Lambda(f - g)(\tau_\bullet\psi_n^R) = \psi_n * (f - g) \to \delta_0 * (f - g) = f - g$ in L^1_{loc}, hence $f = g$ almost everywhere. □

Example 4.14. *δ is not a function! I.e. its not of the form Λf for some $f \in L^1_{loc}$*

Proof. We have $\Delta|_{\Omega\setminus\{0\}} = \Lambda 0$ so if it would be a function, then it would be zero almost everywhere, hence 0. But this is a contradiction. □

Corollary 4.15. *Then $C^\infty(\mathbb{R}^d)$ is dense in $\mathcal{D}'(\mathbb{R}^d)$.*

Proof. We know by 4.12 that there exists $\Lambda\psi_n \to \delta_0$ in $\mathcal{D}'$. Let F be a distribution on $\mathbb{R}^d$. Now setting $F_n := F * \psi_n^R$, yields pointwise

$$F_n(\phi) = F(\psi_n * \phi) \to F(\delta_0 * \phi) = F(\phi)$$

by 4.11 hence $F_n \to F$ in $\mathcal{D}'$.

□

4.1.2 Derivatives

For $\phi, \psi \in D$ we have

$$\int_\Omega \partial^\alpha \phi \psi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \phi \partial^\alpha \psi \,\mathrm{d}x$$

This motivates the definition

Definition 4.16. *For a multiindex α and a distribution F define the distribution*

$$\partial^\alpha F(\phi) = (-1)^{|\alpha|}(F\partial^\alpha \phi)$$

Remark 4.17. *let $f \in L^1_{loc}(\Omega)$. If there exists some $f' \in L^1_{loc}(\Omega)$, such that $\Lambda f' = \partial^\alpha \Lambda f$ as distributions, then we call f' the* **weak derivative** *of f with respect to α.*

Proposition 4.18. *For $F \in D', \phi \in D$, We have*

$$\partial^\alpha (F * \phi) = (\partial^\alpha F) * \phi = F * \partial^\alpha \phi$$

Proof. First note, that holds in the case where F is a test function, so that we have ordinary convolution. Then check pointwise. After erasing the sign $(-1)^{|\alpha|}$

$$F(\phi^R * \partial^\alpha \psi) = F(\partial^\alpha(\phi^R * \psi)) = F((\partial^\alpha \phi) * \psi)$$

□

4.1.3 Support

The Support of a distribution F is the complement of the largest open subset U, such that $F(\phi) = 0 \forall \phi \in D, \mathrm{Supp}(\phi) \subset U$. This definition is unambiguous: If F vanishes on each U_i for some index set I and $\phi \in D$ such that the compact set $\mathrm{Supp}\,\phi \subset U = \bigcup U_i$, we may assume that I is finite and choose a partition of unity $\mathrm{Supp}\,\eta_i \subset U_i$ and $\sum \eta_i = 1$. Then $F(\phi) = \sum F(\phi\eta_i) = 0$.

4.1.4 Tempered Distributions

We no enlarge our test space to the schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ consisting of smooth functions that are rapidly decreasing at ∞ with all derivatives.

Definition 4.19. *Consider the increasing sequence of norms on $C^\infty(\Omega)$ defined by*

$$\|\phi\|_N = \sup\{|x^\beta(\partial_x^\alpha \phi(x))| \mid x \in \mathbb{R}^d, |\alpha|, \|\beta\| \leq N\}$$
$$\mathcal{S} = \{\phi \in C^\infty(\mathbb{R}^d) \mid \|\phi\|_N < \infty \forall N\}$$

with the obvious notion of convergence.

Lemma 4.20. *We have a continuous inclusions $\mathcal{D} \subset \mathcal{S}$, hence $\mathcal{S}' \subset \mathcal{D}'(\mathbb{R}^d)$.*
Moreover, this inclusion is dense. Hence being tempered is a property of distributions.

Lemma 4.21. *Let $f \in L^1_{loc}(\mathbb{R}^d)$ such that there exists $N \geq 0$ with*

$$\int_{|x|<R} |f(x)| \, \mathrm{d}x = O(R^N), \text{ as } R \to \infty$$

The distribution Λ_f is tempered.

Example 4.22. *This condition holds for functions in $L^p(\mathbb{R}^d)$ for $p \in [1, \infty]$*

Lemma 4.23. *If $F \in \mathcal{D}'$ has compact support then it is tempered: Choose $\eta \in \mathcal{D}$ such that $\eta|_U = 1$ on some neighborhood $U \supset \operatorname{Supp}(F)$. Then $F(\eta\phi) = F(\phi)$ for all $\phi \in \mathcal{D}$ and $\phi \mapsto F(\eta\phi)$ defines a continuous functional on $\mathcal{S}$. It does not depend on the choice of η, because given another such η' and any $\phi \in \mathcal{S}$ we have $\operatorname{Supp}((\eta - \eta') \cdot \phi) \cap \operatorname{Supp}(F) = \varnothing$.*

Let F denote a tempered distribution.

Lemma 4.24.

- *All $\partial^\alpha F$ are tempered.*
- *Let $\psi \in C^\infty$ be slowly increasing, i.e. for each α exists N_α such that $\partial_x^\alpha \psi(x) = O(|x|^{N_\alpha})$. Then ψF, defined by $(\psi F)(\phi)(F(\psi\phi)$ is tempered.*

Example 4.25. *If $\psi \in \mathcal{S}$, then $F(\tau_\bullet(\psi^R))$ is slowly increasing. The other formulation is still valid because $\mathcal{S}$ is stable under $*$.*

What is the point of the Schwartz space?

Definition 4.26. *The fourier transformation is a continuous bijection*

$$\mathcal{S} \to \mathcal{S}$$
$$\phi \mapsto \hat{\phi} = (\xi \mapsto \int_{\mathbb{R}^d} \phi(x) e^{-2\pi i x \xi} \, \mathrm{d}x)$$

Lemma 4.27. *We have*

$$\Lambda_{\hat{\psi}}(\phi) = \int_{\mathbb{R}^d} \hat{\psi}(x)\phi(x) \, \mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\hat{\phi}(x) \, \mathrm{d}x = \Lambda_\psi(\hat{\phi})$$

So its easy to define a compatible generalization to the tempered distributions:

Definition 4.28. *Define*

$$\hat{F}(\phi) = F(\hat{\phi})$$

and similarly for the inverse transform $f \mapsto \check{f}$.

We automatically have the inversion theorem for distributions, because it holds for test functions.

Example 4.29. *If $1 \in S$ denotes the constant function at 1, then*

$$\hat{\delta} = \Lambda_1$$

because

$$\hat{\delta}(\phi) = \hat{\phi}(0) = \int 1\phi(x) \, \mathrm{d}x = \Lambda_1(\phi)$$

4.2 Fundamental solutions

In this chapter we may omit the Λ. Fix a partial differential operator L

$$L = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha \text{ on } \mathbb{R}^d$$

with $a_\alpha \in \mathbb{C}$.

Definition 4.30. *A fundamental solution of L is a distribution F such that $L(F) = \delta$*

The reason why this is interesting:

Lemma 4.31. *The operator*

$$f \mapsto T(f) := F * f$$

defines an inverse to L

Proof. because

$$\partial^\alpha(F * f) = (\partial^\alpha F) * f = F * (\partial^\alpha f)$$

Summing over α gives

$$L(F * f) = \delta * f = F * Lf$$

Now recall $\delta * f = f$. □

Definition 4.32. *The characteristic polynomial of L is*

$$P(\xi) = \sum_{|\alpha|\leq m} a_\alpha (2\pi i \xi)^\alpha$$

It is defined it such a way, that $\widehat{Lf} = P \cdot \hat{f}$. So our hope is

$$F := 1/\widetilde{(P(\xi))} = \int \frac{1}{P(\xi)} e^{2\pi i x \xi} \, \mathrm{d}\xi$$

The problem is, that the zeros of P result in difficulties to define F, even as a distribution.

4.2.1 Laplacian

If $L = \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial^2 x_i}$ So $1/P(\xi) = 1/(-4\pi^2|\xi|^2)$ which lies in L^1_{loc} for $d \geq 3$. To calculate our distribution the following is helpful

Theorem 4.33. *For $\lambda > -d$, Let H_λ be the tempered distribution associated to $|x|^\lambda \in L^1_{loc}$. If $-d < \lambda < 0$ then*

$$\widehat{H_\lambda} = c_\lambda H_{-d-\lambda}$$

with

$$c_\lambda = \frac{\Gamma((d+\lambda)/2)}{\Gamma(\lambda/2)} \pi^{-d/2-\lambda}$$

Set $\lambda := -d + 2$. Hence we can find an appropriate constant C_d such that $F(x) = C_d|x|^{-d+2}$ is a fundamental solution, because $\hat{F}(\xi) = 1/(-4\pi^2|\xi|^2)$. So writing out $\widehat{\Delta F} = 1$.

Proposition 4.34. *If $d = 2$, the function $F := 1/(2\pi)\log|x| \in L^1_{loc}$ is a fundamental solution of Δ*

Proof. Sketch. One can actually compute $\hat{F} = -1/(4\pi^2)\left[\frac{1}{|x|^2}\right] - c'\delta$ for some constant c', where $\left[\frac{1}{|x|^2}\right]$ is a distribution, that replaces the (non locally integrable) function $1/|x|^2$ in an appropriate way:

$$\left[\frac{1}{|x|^2}\right] = \int_{|x|\leq 1} \frac{\phi(x)-\phi(0)}{|x|^2}\,\mathrm{d}x + \int_{|x|>1} \frac{\phi(x)}{|x|^2}\,\mathrm{d}x$$

Notice, that on the complement of zero, this distribution coincides with $1/|x|^2$.

Notation 2. *If $\phi \in C^\infty(\Omega)$ slowly increasing (i.e. all derivatives are bounded by polynomials), define $\phi F \in \mathcal{S}'$ as $\phi F(\psi) = F(\phi\psi)$.*

$$\begin{aligned}\widehat{\Delta F} &= -4\pi^2|x|^2\hat{F} \\ &= |x|^2\left[\frac{1}{|x|^2}\right] - 4\pi^2 c' \underbrace{|x|^2\delta}_{=0} \\ &= 1\end{aligned}$$

□

4.2.2 Heat operator

$L = \frac{\partial}{\partial t} - \Delta_x$ taken over $(x,t) \in \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ i.e. we want to solve the homogeneous initial value problem

$$\begin{cases} L(u) = 0 & ,t>0 \\ u(x,0) = f(x) & ,t=0 \end{cases}$$

for some initial value $f \in \mathcal{S}$.
We have

$$(\frac{\partial}{\partial t}\hat{\mathcal{H}}_t)(\xi) = \widehat{\frac{\partial}{\partial t}\mathcal{H}_t}(\xi) = \widehat{\Delta_x \mathcal{H}_t}(\xi) = -4\pi^2|\xi|^2\hat{\mathcal{H}}_t(\xi)$$

and this is obviously solved by $\mathcal{H}_t = e^{-4\pi^2|\xi|^2 t}$. We may call this the heat kernel

$$\hat{\mathcal{H}}_t(\xi) = e^{-4\pi^2|\xi|^2 t}$$

Note that for $t = 0$, we have $\hat{\mathcal{H}}_0 = 1$, hence $\mathcal{H}_0 = \delta$, so $u(x,t) = (\mathcal{H} * f)(x)$ solves the equation $L(u) = 0$ and $u(x,t) \to f(x)$ in $\mathcal{S}$ as $t \to 0$.

Remark 4.35. *$\mathcal{H}_t \to \delta$ in $\mathcal{S}'$ as $t \to 0$ and $\int_{\mathbb{R}^d} \mathcal{H}_t(x)\,\mathrm{d}x = 1$ for all t*

Now define

$$F(x,t) := \begin{cases} \mathcal{H}_t(x) & , \text{ if } t>0, \\ 0 & ,t\leq 0 \end{cases}$$

F is locally integrable on $\mathbb{R}^{d+1}$ and it actually holds

$$\int_{|t|\leq R}\int_{\mathbb{R}^d} F(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq R$$

so F defines a tempered distribution by 4.21.

Theorem 4.36. *F is a fundamental solution of $L = \frac{\partial}{\partial t} - \Delta_x$.*

Proof. Denote $L' = -\frac{\partial}{\partial t} - \Delta_x$, then we have to see the last equation

$$LF(\phi) = F(L'(\phi)) = \lim_{\varepsilon \to 0} \int_{t \geq \varepsilon} \int_{\mathbb{R}^d} F(x,t)(-\frac{\partial}{\partial t} - \Delta_x)\phi(x,t) \, \mathrm{d}x \, \mathrm{d}t \overset{!}{=} \delta(\phi)$$

Integration by parts

$$\begin{aligned}
&\int_{t \geq \varepsilon} \int_{\mathbb{R}^d} F(x,t)(-\frac{\partial}{\partial t} - \Delta_x)\phi(x,t) \, \mathrm{d}x \, \mathrm{d}t \\
&= - \int_{\mathbb{R}^d} \left(\int_{t \geq \varepsilon} \mathcal{H}_t \frac{\partial}{\partial t} + (\Delta_x \mathcal{H}_t)\phi \, \mathrm{d}t \right) \, \mathrm{d}x \\
&= - \int_{\mathbb{R}^d} \left(\int_{t \geq \varepsilon} \mathcal{H}_t \frac{\partial}{\partial t} + (\frac{\partial}{\partial t} \mathcal{H}_t)\phi \, \mathrm{d}t \right) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathcal{H}_\varepsilon(x)\phi(x,\varepsilon) \, \mathrm{d}x && | \; |\phi(x,\varepsilon) - \phi(x,0)| \leq O(\varepsilon) \text{ uniformly in } x \\
&= \int_{\mathbb{R}^d} \mathcal{H}_\varepsilon(x)(\phi(x,0) + O(\varepsilon)) \, \mathrm{d}x && 4.35 \\
&\to \phi(0,0)
\end{aligned}$$

where in the last line we let $\varepsilon \to 0$ □